United sight to an algebraic operations and convergence

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Abstract. Algebraic operations and algebraic spaces was invented to deal with various algebraic problems. Nowadays we can say that was topologization of algebra. The theory of cathegories became an abstract theory, and classical algebraic spaces became only an examples of simplest convergence spaces. In this article we deal with convergences as instances of multivalued appointments and define the continuity as property of commuting squares. This is beginning point for axiomatisation of multivalued mappings continuity properties.

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1. Algebras and their homomorphisms

1.1 *Algebra* $\mathcal{A}$ will be understood as the carrying set $\mathcal{A}$ with some operations. The *algebraic operation* on the set $\mathcal{A}$ defined by Birkhoff [2] defined is any function over the set of n-tuples $f : \mathcal{A}^n \longrightarrow \mathcal{A}$. The set of operations names Σ is called the *signature of algebra*.

Example. *In the space X the potential set $\mathcal{A} = 2^X$ is defined as set of all subsets $A \subset X$. It can be considered as Boole algebra with such operations:*

The least element $0 : 1 \longrightarrow \mathcal{A}$ is defined as a function from the singlepoint set $1 = \{0\}$ to the set of all subsets 2^X, appointing the wide set $\emptyset \in 2^X$ for the unique point $0 \in 1$.

The largest element $t : 1 \longrightarrow \mathcal{A}$ is defined as a function appointing the whole space $X \in 2^X$ for the unique point $0 \in 1$.

The complement $c : \mathcal{A} \longrightarrow \mathcal{A}$ is defined as a function, appointing the complement $A^c = X \setminus A$ for every partial set $A \subset \mathcal{A}$.

The intersection is defined as a function $\wedge : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, appointing the set of common points $A \cap B$ for every pair of partial sets $\langle A, B\rangle$.

The union is defined as a function $\vee : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ appointing the set of points belonging to either set $A \cup B$ for every pair of partial sets $\langle A, B\rangle$. $\square$

A σ-algebra $\mathcal{A}$ will be an example of nonclassical algebra. In this algebra we have countable operations $\mathcal{A}^{\mathcal{N}} \to \mathcal{A}$. Usually such operations are called as convergence of countable sequences.

1.2 The application $f : X \longrightarrow Y$ will be called *morphism* from the set X to the set Y. The first set X will be called a *source space*, and the second set Y will be called a *target space* of morphism. For the point in the source space $x \in X$ the application appoints the point in the target space $f(x) \in Y$, sometimes we denote

$$y = f(x) \uparrow x \in X \ .$$

We have the law of composition for two application with *intermediant space*, i.e. the target space of the first application $f : X \longrightarrow Y$ coincides with the source space of second application $g : Y \longrightarrow Z$. The result will be $f \circ g : X \longrightarrow Z$ an application from the source space of the first application to the target space of the second application. For the point $x \in X$ it appoints the point $z \in Z$ which is calculated

$$z = (f \circ g)(x) = g(f(x)) \ .$$

For every set X we define the identity application $\mathtt{Id}_X : X \longrightarrow X$ which for the point $x \in X$ appoints the same point. It will be neutral for the law of composition, i.e. the identity application doesn't change such product, if this product is possible

$$\mathtt{Id}_X \circ f = f = f \circ \mathtt{Id}_Y \ .$$

Bijective application $f : X \longrightarrow Y$ will have an inverse application $g : Y \longrightarrow X$ for the law of composition

$$f \circ g = \mathtt{id}_X \ , \quad g \circ f = \mathtt{Id}_Y \ .$$

It will be called *isomorphism* of sets X and Y.

1.3 We remind the construction of set product. At first we define the *two sets multiplication.* The multiplication of two sets X and Y is defined as the set of ordered pairs

$$X \times Y = \{\langle a1, a2\rangle : a1 \in X, a2 \in Y\}\ .$$

Two sets multiplication is also defined for morphisms. For any pair of sets morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow W$ we define a morphism between sets multiplications

$$f \times g : X \times Y \longrightarrow Z \times W$$

which appoints

$$(f \times g)(\langle x, y\rangle) = \langle f(x), g(y)\rangle \in Z \times W \uparrow \langle x, y\rangle \in X \times Y\ .$$

Such multiplication maintains the identity morphisms

$$(\mathtt{Id}_X \times \mathtt{Id}_Y) = \mathtt{Id}_{X\times Y}$$

and the composition of morphisms, i.e. for the pairs of composable morphisms $\langle f_1, f_2\rangle$ and $\langle g_1, g_2\rangle$ we have the pair of composable products $\langle f_1 \times g_1, f_2 \times g_2\rangle$ and equality

$$(f_1 \circ g_1) \circ (f_2 \times g_2) = (f_1 \times g_1) \circ (f_2 \times g_2)\ .$$

Therefore the two sets mwltiplication can be concidered as bifunctor.

The sets product can be generated by two sets multiplication. The n-tuple $\langle x1, x2, \ldots, xn\rangle \in \mathcal{A}^n$ can be considered as vector with n components, i.e. the mapping from index set $[n]$ to the space of values $\mathcal{A}$. Any multiplication of n sets A_1, A_2, …, A_n can be identified with the set of n-tuples with only one isomorphism of sets. Therefore any two multiplications of the same sets are identified with *distinguished isomorphisms* of sets, i.e. we indicate unique one from possible isomorphisms, which identifies taken two multiplications. The sets product is defined as *abstract notion* in the category of sets, and sets multiplication will be only *concret presentation* of such abstract notion.

The singlepoint set $I = \{*\}$ is neutral for set multiplication, i. e for this set we have distinguished isomorphisms $l_X : I \times X \longrightarrow X$ and $r_X : X \times I \longrightarrow X$ which appoint

$$l_X(\langle *, x\rangle) = x\ , \quad r_X(\langle x, *\rangle) = x \uparrow x \in X\ .$$

These isomorphisms are *natural* for morphisms of sets $f : X \longrightarrow Y$, i.e. we have the commuting diagrams

$$\begin{tikzcd} I \times X \arrow[r, "l_X"] \arrow[d, "1\times f"'] & X \arrow[d, "f"] \\ I \times Y \arrow[r, "l_Y"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \times I \arrow[r, "r_X"] \arrow[d, "f\times 1"'] & X \arrow[d, "f"] \\ Y \times I \arrow[r, "r_Y"'] & Y \end{tikzcd}$$

Also we have coinciding distinguished isomorphisms for one point set $I \times I \longrightarrow I$,

$$l_I(\langle *, *\rangle) = * = r_I(\langle *, *\rangle)\ .$$

Next we indicate the distinguished isomorphism $a_{XYZ} : (X \times Y) \times Z \longrightarrow X \times (Y \times Z)$ defining the associative equality

$$a_{XYZ}(\langle\langle x, y\rangle, z\rangle) = \langle x, \langle y, z\rangle\rangle \uparrow x \in X, y \in Y, z \in Z\ .$$

They also are natural for morphisms of sets, i.e. for morphisms $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$, $h : Z \longrightarrow Z'$ we get commuting diagrams

$$\begin{tikzcd} (X \times Y) \times Z \arrow[r, "a_{XYZ}"] \arrow[d, "(f\times g)"'] & X \times (Y \times Z) \arrow[d, "f\times(g\times h)"] \\ (X' \times Y') \times Z' \arrow[r, "a_{X'Y'Z'}"'] & X' \times (Y' \times Z') \end{tikzcd}$$

Such distinguished izomorphisms we got by identification of taken multiplications with the space of triples $X \times Y \times Z$. Similar identifications also help to prove the pentagonal diagram identity

$$\begin{tikzcd}[column sep=small]
 & (X \times Y) \times (Z \times W) \arrow[dr, "a"] & \\
X \times (Y \times (Z \times W)) \arrow[ur, "a"] \arrow[d, "1\times a"] & & ((X \times Y) \times Z) \times W \\
(X \times (Y \times Z)) \times W \arrow[rr, "a"'] & & X \times ((Y \times Z) \times W) \arrow[u, "a\times 1"']
\end{tikzcd}$$

In Mac Lane book [11] the coherence theorems asserts that we can provide the distinguished isomorphisms for arbitrary finite sets multiplications defined by different binary trees, if these trees according with properties of associative monoid results the same word. I put so much attention to such usual product of sets, because it will be an explicit example for another more interesting products.

If we add the distinguished isomorphisms for twisting

$$t_{XY} : X \times Y \longrightarrow Y \times X \ , \quad t_{X,Y}(\langle x, y\rangle) = \langle y, x\rangle \uparrow x \in X, y \in Y \ ,$$

our product becomes commuting one, and we can get all properties of commutative monoid. The realization of product with the set of pairs remains the same, but now we identify two symmetric products with the help of twisting isomorphisms.

Otherwise we can deal the product of sets without identifying what singlepoint set will be neutral. Then for the product of sets we get the properties of the semigroup, without distinguished isomorphisms l_X and r_X.

1.4 Next we see see another special properties of set product. Every concret realization of the set product will be coadjoint functor to diagonal one.

Diagonal functor for every set X apoints the couple of sets $< X, X >$, and for ervery morphism $f : X \to Y$ appoints the couple of morphisms $< f, f >$. For noncommuting product we take these couples noncommuting

$$< f, g > \neq < g, f > \ .$$

For every set X we define the diagonal morphism

$$i_X : X \longrightarrow X \times X \ , \quad i_X(x) = \langle x, x\rangle \uparrow x \in X \ ,$$

and for pair of sets $\langle Y, Z\rangle$ we define the projections of product

$$p1_{\langle Y,Z\rangle} : Y \times Z \longrightarrow Y \ , \quad p1_{\langle Y,Z\rangle}(\langle y, z\rangle) = y \uparrow \langle y, z\rangle \in Y \times Z \ ,$$

$$p2_{\langle Y,Z\rangle} : Y \times Z \longrightarrow Z \ , \quad p2_{\langle Y,Z\rangle}(\langle y, z\rangle) = z \uparrow \langle y, z\rangle \in Y \times Z \ .$$

Such morphisms are natural for set morphisms, i.e. for morphisms $f : X' \longrightarrow X$, $g : Y \longrightarrow Y'$, $h : Z \longrightarrow Z'$ we have commuting diagrams:

$$\begin{tikzcd}
X \arrow[r, "i_X"] & X \times X \\
Y \arrow[u, "f"] \arrow[r, "i_Y"'] & Y \times Y \arrow[u, "f\times f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \times Z \arrow[r, "p1_{\langle Y,Z\rangle}"] \arrow[d, "g\times h"'] & Y \arrow[d, "g"] \\
Y' \times Z' \arrow[r, "p1_{\langle Y',Z'\rangle}"'] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \times Z \arrow[r, "p2_{\langle Y,Z\rangle}"] \arrow[d, "g\times h"'] & Z \arrow[d, "h"] \\
Y' \times Z' \arrow[r, "p2_{\langle Y',Z'\rangle}"'] & Z'
\end{tikzcd}$$

The first natural morphism i_X defines mapping over set of all morphism pairs

$$\phi_{X\langle Y,Z\rangle} : \langle (X, Y), (X, Z)\rangle \longrightarrow (X, Y \times Z) \ ,$$

which for the pair of morphism $f : X \longrightarrow Y$, $g : X \longrightarrow Z$ appoints composition of morphisms $i_X \circ f \times g : X \longrightarrow Y \times Z$. The second natural morphism pair $\langle p1_{\langle Y,Z\rangle}, p2_{\langle Y,Z\rangle}\rangle$ defines mapping over the set of morphisms

$$\psi_{X\langle Y,Z\rangle} : (X, Y \times Z) \longrightarrow \langle (X, Y), (X, Z)\rangle \ .$$

These mappings are natural for the source and target turnings of morphisms, i.e. for earlier morphisms we have commuting diagrams of mappings

$$\begin{array}{ccc}
\langle\langle X,Y\rangle,\langle X,Z\rangle\rangle & \xrightarrow{\phi_{X\langle Y,Z\rangle}} & \langle X,Y\times Z\rangle \\
{\scriptstyle\langle\langle f,g\rangle,\langle f,h\rangle\rangle}\downarrow & & \downarrow{\scriptstyle\langle f,g\times h\rangle} \\
\langle\langle X',Y'\rangle,\langle X',Z'\rangle\rangle & \xrightarrow[\phi_{X'\langle Y',Z'\rangle}]{} & \langle X',Y'\times Y'\rangle
\end{array}
\qquad
\begin{array}{ccc}
\langle X,Y\times Z\rangle & \xrightarrow{\psi_{X\langle Y,Z\rangle}} & \langle\langle X,Y\rangle,\langle X,Z\rangle\rangle \\
{\scriptstyle\langle f,g\times h\rangle}\downarrow & & \downarrow{\scriptstyle\langle\langle f,g\rangle,\langle f,h\rangle\rangle} \\
\langle X',Y'\times Y'\rangle & \xrightarrow[\psi_{X'\langle Y',Z'\rangle}]{} & \langle\langle X',Y'\rangle,\langle X',Z'\rangle\rangle
\end{array}$$

The natural mappings $\phi_{X,\langle Y Z\rangle}$ will define *adjunction*, and $\psi_{X,\langle Y Z\rangle}$ will define *coadjunction*.

The pair of identities for natural morphisms i_X and $\langle p1_{\langle Y,Z\rangle}, p2_{\langle Y,Z\rangle}\rangle$

$$i_X \circ p1_{\langle X,X\rangle} = \mathtt{Id}_X \ , \quad i_X \circ p2_{\langle X,X\rangle} = \mathtt{Id}_X$$

provides the *first identity of duality*

$$\phi_{X\langle Y,Z\rangle} \circ \psi_{X\langle Y,Z\rangle} = \mathtt{Id}_{\langle\langle X,Y\rangle,\langle X,Z\rangle\rangle} \ ,$$

and the identity

$$i_{Y\times Z} \circ p1_{YZ} \times p2_{\langle Y,Z\rangle} = \mathtt{Id}_{Y\times Z}$$

provides the *second identity of duality*

$$\psi_{X\langle Y,Z\rangle} \circ \phi_{X\langle Y,Z\rangle} = \mathtt{Id}_{\langle X,Y\times Z\rangle} \ .$$

In the case of such identities adjunction and coadjunction are reciprocal isomorphic mappings. Any product realization with such property is isomorphic with identification of unique isomorphism. This can be consequence of the theory for the *dual functors*. We conclude that the multiplication of sets is presentation of unique abstract product. This abstract product is called *Cartesian product* of the sets. Every concrete product will be an *implementation* of such abstract product. The abstract product is identified with possible unique change of implementation. The commuting abstract product of the sets has more changes of implementation, therefore it is more abstract than noncommuting one. Freyd and Scedrov [5] use the commuting Cartesian product, Mac Lane in [11] has'nt understood what abstract categorical notions mean, and he makes no explicit differences between various notions of Cartesian products. Commuting abstract product can be comfortably defined as limit of discrete diagram.

For abstract product we have much more isomorphic implementations. For example we can identify arbitrary singlepoint set with I using the unique isomorphism $f : \{x\} \longrightarrow \{*\}$ which appoints $f(x) = *$, therefore such singlepoint set $\{x\}$ becomes neutral for such abstract set product implementation.

The abstract commuting Cartesian product extends the operator of intersection in Lattice theory and for it the same sign $X \wedge Y$ can be applyed.

1.5 For more complex algebraic operations many sorted algebra can be defined [9]. Let the carrying sets are indexed by the points from sort set $s \in S$ and the name of operation indicates what sort of carrying spaces must be taken, i.e. the signature Σ is indexed over the space $S^* \times S$, where S^* denotes the set of all finite words of alphabet S. Therefore every operation will be a function

$$f_{s1,s2,\ldots,sn;s} : \mathcal{A}_{s1} \times \mathcal{A}_{s2} \times \ldots \times \mathcal{A}_{sn} \longrightarrow \mathcal{A}_s \ .$$

Example. *A directory $\mathcal{C} : \mathcal{O} \longrightarrow \mathcal{O}'$ is understood as a set of arrows from some point in the first set $\mathcal{O}$ to a point in the second set $\mathcal{O}'$. Points in the sets $\mathcal{O}$, $\mathcal{O}'$ are understood as vertexes of arrows. The arrows between two vertexes $X \in \mathcal{O}$, $Y \in \mathcal{O}'$ compound an arrows set $\mathcal{C}(X;Y)$. The first vertex X is called a source of arrow, the second vertex Y is called a target of arrow. A source is an appointment from the set of arrows to the set of vertexes $s : \mathcal{C} \longrightarrow \mathcal{O}$, for each arrow it appoints the source vertex of taken arrow. The target is an appointment from the set of arrows to the set of vertexes $t : \mathcal{C} \longrightarrow \mathcal{O}'$, for each arrow it appoints the target vertex of taken arrow.*

The directory is a sorted algebra with sort set 3

$$\mathcal{A}_1 := \mathcal{A} \ , \quad \mathcal{A}_2 := \mathcal{O} \ , \quad \mathcal{A}_3 := \mathcal{O} \ ,$$

and two operations $f_{1;2} = s : \mathcal{A} \longrightarrow \mathcal{O}$ and $f_{1;3} = t : \mathcal{A} \longrightarrow \mathcal{O}$.

The category $\mathcal{C}$ will be a dirrectory with the same source and target set $\mathcal{O}$. Additionally we must choose the unit arrows $1 \in (x, x)$ and define the arrows composition

$$f \circ g \in (x, z) \uparrow f \in (x, y), g \in (y, z) \ .$$

□

1.6 For more general algebras on the carrying set A we need to chose other interesting products A^n, and then to define algebraic operations $A^n \longrightarrow A$ as morphisms over such product.

Example. *Let we have two instances of directories with some intermediant vertex space $\mathcal{D} : \mathcal{O}_1 \longrightarrow \mathcal{O}_2$ and $\mathcal{D}' : \mathcal{O}_2 \longrightarrow \mathcal{O}_3$. The product of two directories will be a product bundle for the target and source appointments*

$$\mathcal{D} \otimes_{\mathcal{O}_2} \mathcal{D}' : \mathcal{O}_1 \longrightarrow \mathcal{O}_3 \ .$$

It will be a part of product of arrow sets and can be defined as equalizer for target and source appointments, i.e. it will be the set of arrow pairs for which two maps coincide

$$\mathcal{D} \times_{\mathcal{O}_2} \mathcal{D}' = (t = s') = \{\langle f, g\rangle \in \mathcal{D} \times \mathcal{D}' : t(f) = s'(g)\} \ .$$

This new directory product *get distinguished isomorphisms from semilattice of set product. This isomorphisms are commuting with identity morphisms of factor spaces, therefore the equalize of target and source appointments is maintained, and we can take the trace of distinguished isomorphisms over the set of equalizer. There aren't projections or diagonal morphisms for such directory product.*

The singlepoint set can't be taken as neutral element for directory product. The neutral element will be a set of vertexes. We have the distinguished isomorphisms for equalities

$$l_{\mathcal{D}} : \mathcal{O}_1 \otimes_{\mathcal{O}_1} \mathcal{D} \longrightarrow \mathcal{D} \ , \quad r_{\mathcal{D}} : \mathcal{D} \otimes_{\mathcal{O}_2} \mathcal{O}_2 \longrightarrow \mathcal{D} \ .$$

For the sake of clarity we express the product bundle as the sum of nonintersecting arrow sets

$$\mathcal{D} = \bigcup\{(a, b) : a \in \mathcal{O}_1, b \in \mathcal{O}_2\} \ , \quad \mathcal{D}' = \bigcup\{(a', b') : a' \in \mathcal{O}_2, b' \in \mathcal{O}_3\} \ .$$

The product of two directories is get by the product of slices

$$\mathcal{D} \otimes_{\mathcal{O}_2} \mathcal{D}' = \bigcup\{(a, b) \times (b, c) : a \in \mathcal{O}_1, c \in \mathcal{O}_2\} \ .$$

The set of vertexes $\mathcal{O}$ is understood as directory with identity morphism $\mathtt{Id} : \mathcal{O} \longrightarrow \mathcal{O}$ taken as source and target appointments. There is exactly one arrow over each diagonal pair of vertexes $\langle c, c\rangle$. The products by such directory are defined

$$\mathcal{O}_1 \otimes_{\mathcal{O}_1} \mathcal{D} = \bigcup\{\ \{a\} \times (a, b) : a \in \mathcal{O}_1, b \in \mathcal{O}_2\} \ ,$$

$$\mathcal{D} \otimes_{\mathcal{O}_2} \mathcal{O}_2 = \bigcup\{(a, b) \times \{b\} : a \in \mathcal{O}_1, b \in \mathcal{O}_2\} \ .$$

Therefore the distinguished isomorphisms is defined over slices identifying the set product by singlepoint set with the taken set of arrows

$$\{a\} \times (a, b) \longrightarrow (a, b) \ , \quad (a, b) \times \{b\} \longrightarrow (a, b) \ .$$

□

1.6.1 The set product provides a new algebra with product operations.

Proposition. *If the set product interchange with the product of algebra by isomorphisms*

$$X^n \times Y^n \longrightarrow (X \times Y)^n \ ,$$

then the set product of carrying spaces provides a new algebra with the set product operations.

Proof: It is enough to use the inverse of distinguished isomorphisms

$$I \times I \longrightarrow I \ , \quad X^n \times Y^n \longrightarrow (X \times Y)^n$$

to define the 0-degree and n-degree operations in $X \times X$

$$\begin{array}{ccc} I \times I & \longleftarrow & I \\ {\scriptstyle f_1 \times f_2}\downarrow & & \downarrow \\ X \times Y & \xrightarrow[1]{} & X \times Y \end{array} \qquad \begin{array}{ccc} X^n \times Y^n & \longleftarrow & (X \times Y)^n \\ {\scriptstyle f_1 \times f_2}\downarrow & & \downarrow{\scriptstyle f} \\ X \times Y & \xrightarrow[1]{} & X \times Y \end{array}$$

□

Such algebra will be called a *set product algebra*. In the case when the set product is associative and commuting one, and algebra's operations can be defined over the set product.

For any pair of directories $\mathcal{D}_1$ and $\mathcal{D}_2$ the set product $\mathcal{D}_1 \times \mathcal{D}_2$ provides a new product directory with product operations, i.e. the source and target appointments is defined by products

$$s_1 \times s_2 : \mathcal{D}_1 \times \mathcal{D}_2 \longrightarrow \mathcal{O}_1 \times \mathcal{O}_2 \ , \quad t_1 \times t_2 : \mathcal{D}_1 \times \mathcal{D}_2 \longrightarrow \mathcal{O}'_1 \times \mathcal{O}'_2 \ .$$

Example. *For the categories the set product interchanges with bundle product, used to define the operations of category,*

$$(\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) \times (\mathcal{B} \otimes_{\mathcal{O}'} \mathcal{B}) \longrightarrow (\mathcal{A} \times \mathcal{B}) \otimes_{\mathcal{O} \times \mathcal{O}'} (\mathcal{A} \times \mathcal{B}) \ ,$$

therefore we can define the appointment of unit arrow $\mathcal{O} \times \mathcal{O}' \longrightarrow \mathcal{A} \times \mathcal{B}$ and appointment of arrows composition $(\mathcal{A} \times \mathcal{B}) \otimes_{\mathcal{O} \times \mathcal{O}'} (\mathcal{A} \times \mathcal{B}) \longrightarrow \mathcal{A} \times \mathcal{B}$. The set product of categories provides product category with product operations. □

1.7 For the algebras X and Y with 1-degree operation it is easy to define the sum of such algebras $X + Y$. We take the nonintersecting union of sets $X + Y$ and define the sticked operation as the sum

$$f + f' : X + Y \longrightarrow X + Y \ .$$

For intersecting carriers $X \cap Y \neq \emptyset$ we must check the coincidence of sticked operations over the commune part $X \cap Y$.

For 0-degree operations we need another construction of new operation over the sum of carriers. The same problem will also arise for bigger $d \geq 1$ degree operations. For the sum of groups $X + Y$ we must demand that the units of both groups coincided and decide what would be the composition of members from different sets. The unique solution is to use the product group compounded of pairs $X \times Y = \{\langle x, y\rangle : x \in X, y \in Y\rangle\}$ with the inclusion of summands as partial groups $X \longrightarrow X \times Y$, $Y \longrightarrow X \times Y$ taking pairs with unit of another sumamnd $\langle x, e\rangle \in X \times Y$ and $\langle e, x\rangle \in Y$.

The sum of directories poses no problem. The sum of two directories $\mathcal{D}_1 : \mathcal{O}_1 \longrightarrow \mathcal{O}'_1$ and $\mathcal{D}'_2 : \mathcal{O}_2 \longrightarrow \mathcal{O}'_2$ will have the sum of arrow sets $\mathcal{D}_1 + \mathcal{D}_2$ and will have sourse set the sum $\mathcal{O}_1 + \mathcal{O}_2$ and target set the sum $\mathcal{O}_2 + \mathcal{O}'_2$. Therefore we can define source appointment $\mathcal{D}_1 + \mathcal{D}_2 \longrightarrow \mathcal{O}_1 + \mathcal{O}_2$ and target appointment $\mathcal{D}_1 + \mathcal{D}_2 \longrightarrow \mathcal{O}'_1 + \mathcal{O}'_2$.

The sum of categories $\mathcal{A} + \mathcal{B}$ will have the same arrows as the sum of directories. The unit arrow can be taken in each subcategory separately. The composition is defined only for arrows from one subcategory in order to have common middle vertex. We shall say that taken categories are *nonintermittent subcategories* $\mathcal{A} \subset \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A} + \mathcal{B}$.

1.8 The *semigroup* is an example of algebra with one law of multiplication

$$(\circ) : X \times X \longrightarrow X \ .$$

Such algebra is called *monoid without unit*.

We ask that this multiplication would be associative, i.e. for all points we have identities

$$(a \circ b) \circ c = a \circ (b \circ c) \ .$$

This algebra is not free, we shall say that this algebra satisfies relations.

The *monoid* additionally demands to take the neutral element $e \in X$ with identities

$$x \circ e = x \ , \quad e \circ x = x \uparrow x \in X \ .$$

Such element is understood as operation of 0-degree

$$e : X^0 = I \longrightarrow X \ .$$

The *group* demands to have the inverse element $x^{-1} \in X$ for every point $x \in X$ with two identities

$$x \circ x^{-1} = e = x^{-1} \circ x \uparrow x \in X \ .$$

The inverse element can be understood as operation of 1-degree

$$X \longrightarrow X \ , \quad x^{-1} \uparrow x \in X \ .$$

In the case of existence, such element is unique in associative monoid. For two points y and z wich are inverse elements of x we get equality

$$y = y \circ e = y \circ (x \circ z) = (y \circ x) \circ z = e \circ z = z \ .$$

Therefore the inverse element x^{-1} can be defined as solution of equation in associative monoid

$$- \circ x = e \ , \quad x \circ - = e \ .$$

1.9 The category $\mathcal{C}$ will be a directory with the same source and target vertexes set $\mathcal{O}$ and being associative monoid for the directory product. The multiplication of arrows is called composition

$$(\circ) : \mathcal{C} \otimes_{\mathcal{O}} \mathcal{C} \longrightarrow \mathcal{C} \ ,$$

and neutral element will be appointment of unit arrows

$$1 : \mathcal{O} \longrightarrow \mathcal{C} \ .$$

Group for such directory product has its own name *grupoid*. The semigroup for the directory product is not so usual.

Example. *The class of all directories provides an example of category with taken composition and unit arrows. If we take only directories with finite number of arrows, we shall not have the units in such category. Therefore it can be called the category without units. This provides a serious example of semigroup for the directory product.* □

We can define the category as an algebra with operations being partial mappings over the set product. Such definition is given in Freyd and Scedrov [5] p. 3. Such algebras with partial operations they have called essentially algebraic, in the sense that we can construct special products and define functional operations over these product to get the same (isomorphic) algebra. In general case the partial operations must be deal as arbitrary convergence.

1.10 The morphism between two algebras with the same signature for set product is defined as an application $u : \mathcal{A} \longrightarrow \mathcal{B}$ which maintains the homologic operations, i.e. for each operation name we have the commuting diagram:

$$\begin{tikzcd} \mathcal{A}^n \arrow[r, "u^n"] \arrow[d, "f"'] & \mathcal{B}^n \arrow[d, "f'"] \\ \mathcal{A} \arrow[r, "u"'] & \mathcal{B} \end{tikzcd}$$

Others abstract products are defined in its own categories, and application is changed by arrows between carrying spaces.

Example. *The functor will be morphism of categories $F : \mathcal{A} \Longrightarrow \mathcal{B}$. At first it must be morphism of underlying directory, i.e. we must have the application of arrows $F : \mathcal{A} \longrightarrow \mathcal{B}$ and application of vertexes $F_0 : \mathcal{A}_0 \longrightarrow \mathcal{B}_0$ which must commute with the source and target appointments. Secondly we must demand the maintenance of arrow operations over the directory products. It must maintain the units and the composition of arrows*

$$\begin{tikzcd} \mathcal{O} \arrow[r, "F_0"] \arrow[d, "1"'] & \mathcal{O}' \arrow[d, "1"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A} \arrow[r, "F \otimes_{\mathcal{O}} F"] \arrow[d, "(\circ)"'] & \mathcal{B} \otimes_{\mathcal{O}'} \mathcal{B} \arrow[d, "(\circ)"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{B} \end{tikzcd}$$

□

All set mappings $f : X \longrightarrow Y$ compounds the *category of sets*, we shall note it $\mathtt{Set}$. Vertexes will be all possible (small) sets. The unit arrow will be identity application $\mathtt{Id}_X : X \longrightarrow X$. The arrow composition is taken the usual composition of functions $f \circ g$.

The product category $\mathtt{Set} \times \mathtt{Set}$ has pairs of sets $\langle X, Y \rangle$ as vertexes and pairs of mappings $\langle f : X \longrightarrow Z, g : Y \longrightarrow W \rangle$ as arrows. The bifunctor will be a functor over the product category. The sets multiplication is an example of bifunctor in the category of sets $\mathtt{Set}$

$$(\times) : \mathtt{Set} \times \mathtt{Set} \Longrightarrow \mathtt{Set} \ .$$

1.11 More generally operation in the set category may be arbitrary *multivalued function* $F : \mathcal{A}^n \rightarrow\!\!- \mathcal{A}$. Such algebras are called *multialgebras*. When the operations are arbitrary partial functions, such algebras are called *partial algebras*. If we take the operations over infinite products $\mathcal{A}^{\mathcal{N}}$, such algebras will be better interpreted as convergence space. It is nothing with any finite algebraic property, but traditionally it will be called algebra. Such tradition is maintained also by categorical framework of algebras in Mac Lane's book [11] part VI. The united sight to an algebras calls all algebraic or nonalgebraic operations as convergences and algebras identifies with topological spaces in generalized sense as stuctures defined by Bourbaki [3]

The category was introduced as "abstract nonsense" by S. Eilenberg and S. MacLane in [6]. Their theory emphasizes the work with arrows. On the points of sets constructed "concrete" functions are changed by arrows and many properties of arrows can be proved without investigation of concrete structure of these functions. We can say that a category is one step of abstraction. We use various structure in the category's set of arrows, and the second step of abstraction would be changing this concrete structure in the set of arrows by some other categorical properties. In such way we encounter the *bicategories*. We want emphasize that categorical argument are not absolute fashion of modern mathematics. The "concrete" structures explored by N. Bourbaki must also be admired by "working mathematicians".

The algebras is an example of concrete structure. In contrast the abstract notion of Cartesian product will be an example of abstract notion. This contrast is very clear in computer programming. Algebras will be identifyed with working programs in concrete computer, and categorical abstract notions will be only specification of working programs. Nevertheless the categorical approach made essential progress in todays programming business.

The monoidal category is defined as category with some bifunctor

$$\square : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \ .$$

It was first explicitly mentioned in 1963 by Bénabou [1]. The name "monoidal" is due to Eilenberg.

This bifunctor is called a multiplication and will generalize the set multiplication of the set category. One demands that this multiplication would be associative after identification with a natural transformation of functors

$$\alpha : (-\square-)\,\square- \longrightarrow -\square\,(-\square-) \ .$$

For the coherence one needs that the pentagon diagram with wedges get from natural transformation α would be commuting.

At this time in physics is introduced the premonoidal categories, see W. Joyce [10]. It is interesting that in his work noncommuting pentagon diagram are completed to commuting one with a deformation arrow. The calculating of this arrows provides nonassociative statistics in quantum physics.

1.12 A multivalued mapping between points of two sets is identified with partial set of set multiplication $R \subset X \times Y$. Again the first space is called a source space and the second is called a target space. We shall also say that we have a *reform* from the source space to the target space.

The pair of points $\langle x, y\rangle \in R$ is called *related*. More exactly we shall say that the first point $x \in X$ reforms himself to the second point $y \in Y$ and the second point is reformed from the first one. We can also say that second point is *related* to the first one, and the first point is *corelated* to the second one. It is more usual to say that the second point $y \in Y$ is related to the first point $x \in X$ by the relation R, or the first point $x \in X$ has related point in the target space. But relation has no source or target space, therefore such language can't be correct for educated mathematician.

The reform $A \subset \mathcal{O} \times \mathcal{O}'$ will be an example of directory which has no more than one arrow for every pair of vertexes.

The reform is uniquely defined by its *direct appointment*

$$x \longmapsto x \circ R \ , \quad X \longrightarrow 2^Y \ ,$$

or by its *opposite appointment*

$$y \longmapsto R \circ y \ , \quad Y \longrightarrow 2^X \ .$$

The first appointment to a point $x \in X$ appoints all related points in the target space $x \circ R$. The second appointment to a point $y \in Y$ appoints all the points the source space $x \in X$ to which the taken point is related.

Two reforms $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ can be composed

$$F \circ G := \{\langle x, z\rangle \in X \times Z : (\exists y \in Y)(\langle x, y\rangle \in F, \langle y, z\rangle \in G)\} \ .$$

The identity reform is defined by diagonal in set product

$$\mathtt{Id}_X = \triangle \subset X \times X \ .$$

We can check the associativity for composition of reforms, and all reforms between arbitrary small sets compound a new category which will be called the allegory of sets. It extends the category of sets compounded by all mappings between arbitrary small sets.

The reform $F \subset X \times Y$ is called *entire* if every point in the source space has related point in target space.

The reform $F \subset X \times Y$ is called *accurate* if every point in the source space has no more than one related point in the target space.

Also we rename these properties of opposite reform.

The reform $F \subset X \times Y$ is called *covering* if every point in the target space is related with some point in the source space.

The reform $F \subset X \times Y$ is called *exact* if every point in the target space is related with no more than one point in the source space.

The entire accurate reform is a function, the covering function is *surjection*, the exact function is *injection*.

For a reform $F : X \longrightarrow Y$ between two spaces we construct a function of *direct image* between potential spaces $F^P : 2^X \longrightarrow 2^Y$ defining *image* of arbitrary set in a source space $A \subset X$

$$F^P(A) := \{y \in Y : (\exists x \in A)(\langle x, y\rangle \in F)\} \ ,$$

and a function of *opposite image* $F^{*P} : 2^Y \longrightarrow 2^X$ defining *opposite image* of arbitrary set in a target space $B \subset Y$

$$F^{*P}(B) := \{x \in X : (\exists y \in B)(\langle x, y\rangle \in F)\} \ .$$

For the mappings $f : X \longrightarrow Y$ the opposite image is called an inverse image and is denoted more simply

$$f^{-1}(B) = f^{*P}(B) \ .$$

1.13 The reforms can be abstractly defined in arbitrary regular category, cl. Freyd and Scedrov [5]. Cartesian category has the *beginning* of every finite diagrams. It is enough to have *Cartesian product*, *terminal object* and *equalizer* of every morphism pair.

All these constructions are *abstract*, every object is identified only with *uniquely existent isomorphism.* Regular category additionally demands the existence of *image* for every arrow and the property of product bundle to maintain covers, i.e. for covering base mapping we get covering induced mapping of bunde. The reform $R : X \rightarrow\!\!\!\!\!- Y$ in Cartesian category is defined as *monopair* of *tabulating mappings* $f : Q \longrightarrow X$ and $g : Q \longrightarrow Y$. The relation is understood as part of Cartesian product defined by this monopair. It can be named as *graphic* of reform

$$G(R) \rightarrowtail X \wedge Y \ .$$

The reform R can be restore from it's graphic $G(R) \rightarrowtail X \wedge Y$ by choosing what the space is a source and what the space is a target, i.e. the reform can be denoted by the triple

$$R = \langle X, G(R), Y \rangle \ .$$

In regular category we have possibility to define the composition of two reforms with an intermediate space.

The mapping $f : X \longrightarrow Y$ are identifyed with tabulation $1 : X \longrightarrow X$, $f : X \longrightarrow Y$ and arbitrary reform is equal to fraction of its tabulation $R = f^{-1} \circ g$. All reforms of regular category $\mathcal{C} \subset \mathcal{R}(\mathcal{C})$ compound an allegory. We shall call it a fractional extention. Functor of Cartesian category will be called representation of Cartesian category if it maintains the beginnings of finite diagrams. Functor of regular category will be called representation of regular category if it additionally maintains the covers. Every representation of regular category $F : \mathcal{C} \Longrightarrow \mathcal{C}'$ has fractional extension which becomes a representation of *unitary allegories* $\mathcal{R}(F) : \mathcal{R}(\mathcal{C}) \Longrightarrow \mathcal{R}(\mathcal{C}')$.

In Cartesian category we can freely define the composition of function with arbitrary relation $f \circ R$. A topos will be Cartesian category $\mathcal{C}$ having potential objects. For any object C we define potential $[C]$ with distinguished reform $\ni: [C] \rightarrow\!\!\!\!\!- C$ wich must be initial among reforms to this object by transformations defined with arrows of category, i.e. for every reform $R : X \rightarrow\!\!\!\!\!- C$ we can find a unique representing arrow $f : X \longrightarrow C$ which transforms the distinguished reform to the taken one

$$f \circ \ni = R \ .$$

The set category is an example of topos. The potential of the set C will be the set of all subsets $[C] := 2^C$ and distinguished reform every partial set $A \subset C$ relates with its points $x \in C$, we shall write $A \ni x$.

For the reform $R : X \rightarrow\!\!\!\!\!- C$ the unique representing mapping $f : X \longrightarrow [C]$ is a direct appointment of this relation. For any reform $R : X \rightarrow\!\!\!\!\!- Y$ in topos we can define the unique mapping $R^P : [X] \longrightarrow [Y]$. In the Set category this mapping coincides with direct image appointment. The direct image appointment defines endofunctor of set category $F : \mathtt{Set} \Longrightarrow \mathtt{Set}$. Its fractional extention is representation of unitary allegories $\mathcal{R}(F) : \mathcal{R}(\mathtt{Set}) \Longrightarrow \mathcal{R}(\mathtt{Set})$. We remark that it differs from earlier direct image appointment. For the relation $R : X \rightarrow\!\!\!\!\!- Y$ tabulated with the pair of mappings $\langle f, g \rangle : Q \longrightarrow \langle X, Y \rangle$ the set $A \subset X$ has related set $B \subset Y$ exactly when there is a set $C \subset Q$ with needed direct images for tabulating mappings

$$f^P(C) = A \ , \quad g^P(C) = B \ .$$

We get maintenance for composition $F \circ G$ and inverce F^* of relations

$$\mathcal{R}(F \circ G) = \mathcal{R}(F) \circ \mathcal{R}(G) \ , \quad \mathcal{R}(F^*) = \mathcal{R}(F)^* \ .$$

The direct image $f^P(A) \subset Y$ are related to taken set $A \subset X$ by the trace of set product $(A \times f^P(A)) \cap R \subset X \times Y$. It is the biggest of related sets to taken one.

2. Reform's continuity in convergence spaces

2.1 The operations in multialgebras better to interpret as convergences. Then morphisms of multialgebras will be defined as mappings with various continuity properties.

The *convergence* in the *world space* X is understood as a reform $\rho \subset X^\Phi \times X$ from sequences $\mathcal{F} \in X^\Phi$ to the points $x \in X$ of taken set X, see Gähler [8]. The direct appointment transforms the sequence $\mathcal{F} \in X^\Phi$ to a partial set $\mathcal{F} \circ \rho \subset X$ which is called a *limit* of the taken sequence. We shall use the notation

$$\mathtt{lim}_X \ \mathcal{F} := \mathcal{F} \circ \rho \subset X \ .$$

The points of the limit will be called *limit points*. The opposite appointment transforms the point $x \in X$ to a set of sequences $\rho \circ x \subset X^{\Phi}$. The sequences from this set will be called *converging to the taken point*.

A *countable sequence* in the space X will be a point from the countable set product of the world space $\mathcal{F} \in X^{\mathcal{N}}$, i. e. it is a function over the natural numbers to the world space

$$\mathcal{F}_n \in X \uparrow n \in \mathcal{N} \ .$$

A natural number $n \in \mathcal{N}$ will be called an index and a value $\mathcal{F}_n \in X$ will be called a n-member of the sequence.

The algebraic operations traditionally deals with the convergence of *finite sequences* taken from the finite set product $\mathcal{F} \in X^n$, i.e. a sequence in this case coincides with a n-tuple of space points $x_i \in X$.

The *filter* is a collection of partial sets $\mathcal{F} \subset X^P$. The space of all filters will be the second potential of the space X^{PP}. Filters commonly are considered as generalized sequences.

For us any *set* may also be considered as a sequence. The space of such sequences coincides with potential of the space X^P. The limit point of such convergence may be interpreted as integral of the taken set, for example a geometric center point could be set's limit point

$$x = \int A \ .$$

For any reform to another world space $R : X \to\!\!\!- Y$ we get the reform between the spaces of sequences $R^{\mathcal{N}} : X^{\mathcal{N}} \to\!\!\!- Y^{\mathcal{N}}$ using fractional extension of mappings. Every mapping $f : X \longrightarrow Y$ provides the mapping for the sequences $f^{\mathcal{N}} : X^{\mathcal{N}} \longrightarrow Y^{\mathcal{N}}$ coinciding with set product. For reform we take the fractional extension of set product. It can be expressed in such way: For every sequence with members $x_n \in X$ a sequence with members $y_n \in Y$ will be related exactly then, when these members are related

$$\langle x_n, y_n \rangle \in R \uparrow n \in \mathcal{N} \ .$$

The direct appointment appoints to the sequence $x. \in X^{\mathcal{N}}$ the set coinciding with the product of sets apointed to the members of the sequence

$$\langle x_n : n \in \mathcal{N} \rangle \circ R^{\mathcal{N}} = \prod \{ x_n \circ R : n \in \mathcal{N} \} \ .$$

We describe needed transformations also for another generalized sequences. For the partial sets $A \in X^P$ we take fractional extension of mapping's direct image appointment. Therefore for a partial set $A \subset X$ we get a related partial set $B \subset Y$ in another world space exactly then, when we can find the partial set $C \subset Q \subset X \times Y$ in the graphic set of the reform R which for projections has direct images coinciding with the taken partial sets

$$p_1^P(C) = A \ , \quad p_2^P(B) = B \ .$$

For the filters $\mathcal{F} \subset X^{PP}$ we take fractional extension of mapping's secondary direct image appointment. This secondary image appointment is used to define the image of the filter for a mapping of the world space. For a filter $\mathcal{F} \in X^{PP}$ we have a related filter in another world space $\mathcal{G} \in Y^{PP}$ exactly then, when we can find a filter on the graphic of the reform $\mathcal{H} \in Q^{PP}$ which has the images coinciding with the taken filters

$$p_1^{PP}(\mathcal{H}) = \mathcal{F} \ , \quad p_2^{PP}(\mathcal{H}) = \mathcal{G} \ .$$

The continuous function between two convergence spaces $u : X \longrightarrow Y$ is any function which maintains the limits

$$u^P(\texttt{lim}_X \ \mathcal{F}) \subset \texttt{lim}_Y \ u^{\Phi}(\mathcal{F}) \ .$$

For an algebraic operation, as for function over set product, the limit of (finite) sequence has exactly one limit point. Therefore the continuity for such convergence coincides with equality of limit points. For example, in additive monoid we demand the property of additivity

$$f(x + y) = f(x) + f(y) \ .$$

A map between algebras is called *morphism of algebras* if it is continuous for all algebra's operations.

For example a morphism between semigroups $u : X \longrightarrow Y$ is defined demanding the equality of unique limit points

$$u(\mathcal{F} \circ \rho_X) = u^\Phi(\mathcal{F}) \circ \rho_Y \ .$$

This property is drawn as commuting square

$$\begin{tikzcd}
X^\Phi \arrow[r, "u^\Phi"] \arrow[d, "\rho_X"'] & Y^\Phi \arrow[d, "\rho_Y"] \\
X \arrow[r, "u"'] & Y
\end{tikzcd}$$

We shall say that commuting diagram defines *predicate*. In this case a predicate will be equality between two terms get with composition of arrows

$$\rho_X \circ u = u^\Phi \circ \rho_Y \ .$$

For more general convergence the continuous mappings are defined by by inclusion of limit sets

$$u^P(\mathcal{F} \circ \rho_X) \subset u^\Phi(\mathcal{F}) \circ \rho_Y \ .$$

Such inclusion can be drawn as *directed square*. It can be named as *allegory's commuting square*

$$\begin{tikzcd}
X^\Phi \arrow[r, "u^\Phi"] \arrow[d, "\rho_X"'] & Y^\Phi \arrow[d, "\rho_Y"] \\
2^X \arrow[r, "u^P"'] & 2^Y
\end{tikzcd}$$

We can see that this inclusion is equivalent with another inclusion for opposite reforms

$$u^{-1}(y) \circ \rho_X \subset (u^\Phi)^{-1}(y \circ \rho_Y) \ .$$

It can be drawn by another directed square

$$\begin{tikzcd}
X^\Phi & Y^\Phi \arrow[l, "u^{*\Phi}"'] \\
2^X \arrow[u, "\rho^*_X"] & 2^Y \arrow[l, "u^{*P}"] \arrow[u, "\rho^*_Y"']
\end{tikzcd}$$

We shall say that these two directed squares define two different *reversed predicates.*

2.2 For diagrams of reforms the predicate is defined by choosing the *birth vertex*, the *death vertex*, the two *paths from the birth vertex to the death vertex* and the *arrow of inclusion* between such paths. The path is *cyclic* if it has some repeated vertex. The path is *simple* if it hasn't cyclic parts. It is a unique possibility for simple path to go from the birth vertex to the death vertex. Therefore the predicates with simple paths are determined by choosing the birth and the death vertexes and the direction of inclusion.

If we denote the vertexes of commuting square

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & 2 \arrow[d] \\
4 \arrow[r] & 3
\end{tikzcd}$$

and choose clockwise the positive direction of inclusion, then the former first directed square will be noted (-13) and the former second directed square will be noted $(+31)$. There are 24 predicates with simple paths get from such square diagram. Every of such predicates can be viewed as a definition of some continuity

property. Therefore clever mathematician would wish to choose the 24 different names for every of such property. The linguistic problems usually are not interesting for mathematicians, but there is no other way to understand this new field of multifunctional algebra. For new theories we need a new efforts of richer language. My attempt to propose a vocabulary for englishmen can be corrected by others more prudent wizards of English language.

2.3 At first we shall give only four names initiated by two different definitions of continuous multivalued mappings. Our language will use the notion of sequences and their limit points.

2.3.1 A reform $F : X \rightarrowtail Y$ will be called *respecting* if for every sequence in the source space $\mathcal{F} \in X^\Phi$ we have in the target space the inclusion of points related to the limit points in the source space

$$\mathcal{F} \circ \rho_X \circ F \subset \mathcal{F} \circ F^\Phi \circ \rho_Y \ ,$$

we shall write

$$F^P(\mathtt{lim}_X \mathcal{F}) \subset \mathtt{lim}_Y^P(F^\Phi(\mathcal{F})) \ ,$$

i.e. every point $y \in Y$, related to the limit point $x \in \mathtt{lim}_X \mathcal{F}$, must be a limit point for some related sequence $\mathcal{F}'$. The $\mathtt{lim}_Y^P$ for some set of filters notes the direct image appointment. Such continuity property is noted by predicate (-13). This property expresses the regularity of limit points in the source space. The respective reform can be also understood as distributive operation. In earlier example the additive morphism is distributive for a law of addition.

Such property is useful when we want to construct the algebras having two operations. For example multiplication can be distributive for the operation of sum. Taking all possible products of generaters, we get the set of members which sums maintains the products. Every product of sums will be expressed as a sum of products

$$(x + y) \cdot (z + w) = x \cdot z + y \cdot z + x \cdot w + y \cdot w \ .$$

2.3.2 A reform $F : X \rightarrowtail Y$ will be called *creating* if for every point in the source space $x \in X$ we have in the target space the inclusion of sequences $\mathcal{F}'$ related to convergent sequences $\mathcal{F}$ in the source space

$$x \circ \rho_X^* F^\Phi \subset x \circ F \circ \rho_Y^* \ ,$$

we shall write

$$F^{\Phi P}(\mathtt{lim}_X^*(x)) \subset \mathtt{lim}_Y^{*P}(F(x)) \ ,$$

i.e. every sequence related to a x-convergent sequence must be y-convergent for some point $y \in Y$ related to taken point x. Such property is noted by predicate $(+42)$. This property expresses the existence of limit points in the target space.

2.3.3 These two properties for mappings coincide with usual continuity property: Every limit point in the source space $x \in \mathtt{lim}_X \mathcal{F}$ must have image $f(x) \in Y$ which is a limit point of the image sequence $f(\mathcal{F})$. Reverse predicate will say that the x-convergent sequence $\mathcal{F}$ has the image $f(\mathcal{F})$ convergent to the image $f(x) \in Y$ of the taken point.

2.4 The opposite reform $F^* : Y \rightarrowtail X$ provides additional two predicates. We shall give them new names.

2.4.1 A reform $F : X \rightarrowtail Y$ will be called *reversely cautious* if for every sequence in the target space $\mathcal{F}' \in Y^\Phi$ we have the inclusion of points $x \in X$ having related limits in the target space

$$\mathcal{F}' \circ \rho_Y \circ F^* \subset \mathcal{F}' \circ F^{\Phi *} \circ \rho_X \ ,$$

we shal write

$$(\mathtt{lim}_Y \mathcal{F}') \circ F^* \subset \mathtt{lim}_X^P(\mathcal{F}' \circ F^{\Phi *}) \ ,$$

i.e. every point in the source space $x \in X$, related to the limit point $y \in \mathtt{lim}_Y \mathcal{F}'$ of the taken sequence, must be a limit point for some sequence $\mathcal{F}$ in the source space X for which the taken sequence is related. Such property is noted by predicate $(+24)$. Philologically the convergence in the source space must be cautious as the convergence in the target space could be respecting.

2.4.2 A reform $F : X \rightarrow\!\!\!- Y$ will be called *reversely wasting* if for every point in the target space $y \in Y$ we have inclusion of sequences having related sequences with taken point in the target space

$$y \circ \rho_Y^* \circ F^{\Phi *} \circ \rho_X^* \ ,$$

we shall write

$$(\mathtt{lim}_Y^*(y)) \circ F^{\Phi *} \subset \mathtt{lim}_X^{*P}(y \circ F^*) \ ,$$

i.e. every sequence $\mathcal{F}$ in the source space having related sequence $\mathcal{F}'$ in the target space with taken limit point $y \in \mathtt{lim}_Y \mathcal{F}'$, must be convergent to some point $x \in X$ for which taken point $y \in Y$ is related. Such property is noted by predicate (-31). Philologically the convergence in the source space must be wasting, as the convergence in the target space could be creating.

2.5 The opposition maintains the inclusion of the graphics, therefore we can define additionally 4 predicates with known properties. The predicate $(+31)$ notes the *reversely respecting* reform F, the predicate (-24) notes the *reversely creating* reform F, the predicate (-42) notes the *cautious* reform F, and the predicate $(+13)$ notes the *wasting* reform F. These predicates will define different properties for pairs of conjugate functors. For reforms such functors are given by direct image appointment and opposite image appointment. Therefore we wished to have different names for all predicates. At this moment I choose the predicate with direct reform as principal.

2.6 We have defined the names for 8 predicates. We shall say that these predicates are from the *first octet*. When the reform $F : \rightarrow\!\!\!- Y$ is a mapping, we can drive along this mapping the death or the birth vertexes for the first two predicates (-13), (-24). We get the new predicates (-14) and (-23), which defines the same continuity properties for mappings. These predicates begins the *second octet*.

2.6.1 A reform $F : X \rightarrow\!\!\!- Y$ will be called *lavishing* if for every sequence $\mathcal{F}$ in a source space X every limit point has related point in the target space $y \in Y$ which is a limit point for some related sequence $\mathcal{F}'$

$$\mathcal{F} \circ \rho_X \subset \mathcal{F} \circ F^{\Phi} \circ \rho_Y \circ F^* \ ,$$

we can write

$$\mathtt{lim}_X \mathcal{F} \subset (\mathtt{lim}_Y^P(\mathcal{F} \circ F^{\Phi})) \circ F^* \ .$$

It is more convenient to express such inclusion as large opposite image of convergence in the target space

$$\rho_X \subset (F^{\Phi} \times F)^{*P}(\rho_Y) \ .$$

For the reforms there we apply the fractional extension of set product. Such property is noted by predicate (-14).

2.6.2 A reform $F : X \rightarrow\!\!\!- Y$ will be called *pressing* if for every sequence $\mathcal{F}'$ in a target space Y the limit point $x \in X$ of sequences $\mathcal{F}$ in a source space, which has related taken sequence $\mathcal{F}'$, has related some limit point $y \in \mathtt{lim}_Y \mathcal{F}'$

$$\mathcal{F}' \circ F^{\Phi *} \circ \rho_X \circ F \subset \mathcal{F}' \circ \rho_Y \ ,$$

we shall write

$$(\mathtt{lim}_X^P(\mathcal{F}' \circ F^{\Phi *})) \circ F \subset \mathtt{lim}_Y \mathcal{F}' \ .$$

It can also be expressed that a convergence in a source space has small direct image

$$(F^{\Phi} \times F)^P(\rho_X) \subset \rho_Y \ .$$

Such property is noted by predicate (-23).

2.7 The opposite reform $F^* : Y \rightarrow\!\!\!- X$ provides a new names.

2.7.1 A reform $F : X \rightarrow\!\!\!- Y$ will be called *hiding* if for every sequence $\mathcal{F}'$ in the target space Y every limit point $y \in Y$ is related to some point in the source space $x \in X$, which is a limit point for some sequence $\mathcal{F}$ in a source space, which has the taken sequence $\mathcal{F}'$ as related

$$\mathcal{F}' \circ \rho_Y \subset \mathcal{F}' \circ F^{\Phi *} \circ \rho_X \circ F \ ,$$

we shall write

$$\mathtt{lim}_Y \mathcal{F}' \subset (\mathtt{lim}_X^P(\mathcal{F}' \circ F^{\Phi *})) \circ F \ .$$

It can be expressed that the convergence in a source space has large direct image

$$\rho_Y \subset (F^{\Phi} \times F)^P(\rho_X) \ .$$

Such property is noted by predicate (+23). Philologically the convergence in a source space must be hiding, as the convergence in a target space could be lavishing.

2.7.2 A reform $F : X \to\!\!- Y$ will be called *reflecting* if for every sequence $\mathcal{F}$ in the source space X every point $x \in X$, which has related limit point of some related sequence $\mathcal{F}'$ in a target space, is a limit point of taken sequence $\mathcal{F}$

$$\mathcal{F} \circ F^{\Phi} \circ \rho_Y \circ F^* \subset \mathcal{F} \circ \rho_X \ ,$$

we shall write

$$(\mathtt{lim}_Y^P(\mathcal{F} \circ F^{\Phi})) \circ F^* \subset \mathtt{lim}_X \mathcal{F} \ .$$

It can be expressed that the convergence in a target space has a small opposite image

$$(F^{\phi} \times F)^{*P}(\rho_Y) \subset \rho_X \ .$$

Tis property is noted by predicate (+14).

2.8 The reversing will give the same names for additional 4 predicates. At this moment the principal predicate deals with the limit points of convergence, and reversed predicate deals with convergent sequences of convergence. The predicate (+41) notes the *reversely lavishing* reform F, the predicate (+32) notes the *reversely pressing* reform F, the predicate (−32) notes the *reversely hiding* reform F, the predicate (−41) notes the *opposite reflecting* reform F.

2.9 We have just defined the 8 predicates from the second octet. The rest predicates compound the *third octet*. For functional convergence we can get the earlier continuity properties. This helps us to choose the preference order in description of last predicates.

2.9.1 A reform $F : X \to\!\!- Y$ will be called *thin* if for every sequence $\mathcal{F}$ in a source space X every sequence converging to a point, related to some limit point $x \in X$ of taken sequence $\mathcal{F}$, is related with taken sequence $\mathcal{F}$

$$\mathcal{F} \circ \rho_X \circ F \circ \rho_Y^* \subset \mathcal{F} \circ F^{\Phi} \ ,$$

we shall write

$$\mathtt{lim}_Y^{*P}(F^P(\mathtt{lim}_X \mathcal{F})) \subset F^{\Phi P}(\mathcal{F}) \ .$$

It is better to say that the inverse image of the taken reform is smaller than the reform for the sequences

$$(\rho_X \times \rho_Y)^{*P}(F) \subset F^{\Phi} \ .$$

This property is noted by predicate (−12).

2.9.2 A reform $F : X \to\!\!- Y$ will be called *thick* if for every point $x \in X$ in a source space the limit points in target space $y \in \mathtt{lim}_Y \mathcal{F}'$, of some sequence related to some sequence $\mathcal{F}$ in source space converging to the taken point x, is related to the taken point x

$$x \circ \rho_X^* \circ F^{\Phi} \circ \rho_Y \subset x \circ F \ ,$$

we shall write

$$\mathtt{lim}_Y^P(F^{\Phi P}(\mathtt{lim}_X^*(x)) \subset F(x) \ .$$

It is better to say that the direct image of the reform for sequence is smaller than the taken reform itself

$$(\rho_X \times \rho_Y)^P(F^{\Phi}) \subset F \ .$$

This property is noted by predicate (+43). This property speaks about regularity of existing limit points. Such reforms were often encountered in the theory of differential operators, they are called *operator with closed graphic*.

2.10 The reversing provides additional two names. The predicate (+21) notes the *reversely thin* reform F, and the predicate (−34) notes the *reversely thick* reform F. These predicates deal with graphic of opposite reform R^*.

2.11 Rest 4 last predicates.

2.11.1 A reform $F : X \twoheadrightarrow Y$ will be called *binding* if for every sequence $\mathcal{F}$ in the source space X every related sequence $\mathcal{F}$ in the target space Y has limit point $y \in Y$ related to some limit point $x \in X$ of taken sequence $\mathcal{F}$

$$\mathcal{F} \circ F^\Phi \subset \mathcal{F} \circ \rho_X \circ F \circ \rho_Y^* \ ,$$

we shall write

$$F^\Phi(\mathcal{F}) \subset \mathtt{lim}_Y^{*P}(F^P(\mathtt{lim}_X \mathcal{F})) \ .$$

It is better to say that the inverse image of the taken reform is larger than the reform for sequences

$$F^\Phi \subset (\rho_X \times \rho_Y)^{*P}(F) \ .$$

This property is noted by the predicate (+12).

2.11.2 A reform $F : X \twoheadrightarrow Y$ will be called *parting* if for every point $x \in X$ in a source space X the related point $y \in Y$ in the target space Y must be a limit point of some sequence $\mathcal{F}$ which is related to some sequence $\mathcal{F}$ in a source space X converging to taken point x

$$x \circ F \subset x \circ \rho_X^* \circ F^\Phi \circ \rho_Y \ ,$$

we shall write

$$F(x) \subset \mathtt{lim}_Y^P F^\Phi(\mathtt{lim}_X^*(x)) \ .$$

It is better to say that direct image of the sequence reform is larger than the taken reform itself

$$F \subset (\rho_X \times \rho_Y)^P(F^\Phi) \ .$$

This property is noted by the predicate (−43).

2.12 The reversing provides additional two names. The predicate (−21) notes the *reversely binding* reform F, and the predicate (+34) notes the *reversely parting* reform F. These predicates deals with graphic of opposite reform R^*.

2.13 The reform composition maintains the inclusion

$$F \subset G \Longrightarrow R \circ F \subset R \circ G, F \circ H \subset G \circ H \ .$$

This property gives a possibility to apply the composition of directed squarees. This is straightforward applied for the predicates of first octet.

2.13.1
Proposition. *The composition of reform maintains the property of respecting reforms* (−13)*, creating reforms* (+42)*, cautious reforms* (+24)*, reversely wasting reforms* (+31) *and their reversed counterparts.*
Proof: It is enough to apply composition of *neighbouring squares*. □

2.13.2 The predicates of second octet needs the composition of *embracing squares*.
Proposition. *The composition of reform maintains the property of lavishing reforms* (−14)*, pressing reforms* (−23)*, hiding reforms* (+23)*, reflecting reforms* (+14) *and their reversed counterparts.*
Proof: We construct the embracing squarees. □

2.13.3 For the predicates of third octet we must pose restrictions for the convergence in the mediating space. At first we explain the terminology.

For every reform $R : X \to\!\!\!- Y$ we have defined two appointments: the direct image $R^P : X^P \longrightarrow Y^P$ and inverse image $R^{*P} : Y^P \longrightarrow X^P$. They are monotonic mappings of Boole algebras, therefore we have instances of functors.

We shall call the pair of opposite functors F and G between two categories an *adjunction* with the same notation as for reform $R : \mathcal{A} \to\!\!\!- \mathcal{B}$. Therefore for this adjunction we have the source category $\mathcal{A}$ and the target category $\mathcal{B}$. The direct functor $F : \mathcal{A} \Longrightarrow \mathcal{B}$ is called *adjoint functor*, and opposite functor $G : \mathcal{B} \Longrightarrow \mathcal{A}$ is called *coadjoint functor*. The *unit* of adjunction is a transformation of Identity functor $i : \mathtt{Id} \longrightarrow FG$ and *counit* of adjunction is a transformation of functor composition $e : GF \longrightarrow \mathtt{Id}$. The naturality and triangular identities is needed for the usual *true adjunction*, but we can deal with "generalized" adjunctions without such properties.

The entire reform R provides a unit of its adjunction

$$A \subset R^{*P}(R^P(A)) \uparrow A \subset X \ ,$$

the accurate reform R provides a counit of its adjunction

$$R^P(R^{*P}(A')) \subset A' \uparrow A' \subset Y \ ,$$

the covering reform R provides a unit of opposite adjunction

$$A' \subset R^P(R^{*P}(A')) \uparrow A' \subset Y \ ,$$

and the exact reform R provides counit of opposite adjunction

$$R^{*P}(R^P(A)) \subset A \uparrow A \subset X \ .$$

The mapping $R := f$ provides a unit and counit together, therefore such adjunction for Boole algebras is true. The inverse of mapping $R := f^*$ provides a unit and counit for opposite adjunction, therefore in this case the opposite adjunction is true. For bijective map R both adjunctions are true.

Proposition. *The composition maintains the property of thin reforms* (-12) *if the convergence in the middle space is simple. The composition maintains the property of thick reforms* $(+43)$ *if the convergence in the middle space is entire. The composition maintains the property of binding reforms* $(+12)$ *if the convergence in the middle space is covering. The composition maintains the property of parting reforms* (-43) *if the convergence in the middle space is exact. The reversed counterparts needs the same conditions for the convergence in the middle space.*

Proof: These conditions provide needed unit and co-unit arrows of direct and opposite adjunctions. □

For the countable sequence $x_n \in X \uparrow n \in X$ an accurate convergence is known as Hausdorff. Usual sequence covergence is covering, as it appoints the point $x \in X$ for constant sequence $x_n = x$. The usual convergence is not exact, because different sequences can converge to the same limit point. Entire convergence may be only for some algebraic operation over the finite part of sequences.

2.14 The identity mapping has continuity property of every kind, therefore the respective reforms compound the suballegory. Sometimes it is beneficial to use the decomposition with tabulating mappings $R = f^{-1} \circ g$ in this suballegory, because the continuity properties for mappings is much simple.

The death vertex can be pushed forward along adjoint functor with counit, and backword along adjoint functor with unit. The direction is defined with the arrow of predicate. In the case of set reforms we get that death vertex can be pushed forward along the accurate reform, and can be pushed backward along the entire reform.

The birth vertex can be pushed forward along coadjoint functor with unit, and can be pushed backward along coadjoint functor with counit. In the case of set reforms we get that the birth vertex can be pushed forward along covering reform, and can be pushed backward along exact reform.

After pushing we get the continuity for the new predicate. In the case of mappings we get the equivalent continuity properties. The death vertex can be pushed forward and backward along mappings, and birth vertex can be pushed backward and forward along inverse mappings.

Therefore we can formulate the proposition about continuity properties of mappings.

Proposition. *For mappings 4 continuity properties are equivalent: respecting* (-13), *pressing* (-23), *lavishing* (-14) *and reversely creating* (-24).

Proof: Such predicates allow to push the death and birth vertexes along the mappings or inverse mappings. $\square$

This proposition also is valid for 4 rversed predicates.

3. Reform's continuity in adherence spaces

3.1 The *adherence* in the space X is defined as generalized convergence $\mathtt{Adh} : 2^X \rightarrow\!\!\!- X$. A sequence now will be an arbitrary partial set $A \subset X$. The direct appointment appoints the set of *adherent points*

$$[A] := A \circ \mathtt{Adh}$$

and will be called an *adherence operator*. The opposite appointment appoints the filter of sets having taken adherent point

$$\mathcal{F}_x := \mathtt{Adh} \circ x \ .$$

We shall call such filter an *adherence filter over the point* $x \in X$.

We remember that any of these appointments restore the adherence. We shall say the point $x \in X$ *is adherent* to the set $A \subset X$ and the set *adherent point*. All adherent points for the set $A \subset X$ compound the *adherence set* $[A] \subset A$.

The *conjugate operator* is called *inside operator*, it is defined with the Boole completement of the sets

$$\langle A \rangle := A \circ \mathtt{Ins} = (A^c \circ \mathtt{Adh})^c \ .$$

The sets with the adherent point now will be changed by the neighbourhoods of taken point. The set $V \subset X$ is a neighbourhood of the point $x \in X$ and this point $x \in X$ is inside point of this set exactly then, when the completment of this set $A^c = X \setminus A$ hasn't taken point as adherent point. We shall write

$$\mathcal{V}_x := \mathtt{Ins} \circ x \ .$$

We shall say that the inside points $x \in A$ compound the inside set $\langle A \rangle \subset X$. The inside operator or neighbourhood filters define *inside* as adherence conjugate to taken one.

It is obvious that secondly conjugate operator coincides with the taken one. The names adherence or inside are only relative consideration. At this moment every adherence can be inside of the dual adherence. Next we can ask some asymmetric properties for the adherence.

3.1.1 An *increasing adherence* appoints the bigger adherence set $A \subset [A]$. An *isotonic adherence* for the bigger set apoints a bigger set

$$A \subset B \Longrightarrow [A] \subset [B] \ .$$

The isotonic increasing adherence is called a *closure*. The set is called closed if it coinsides with it's adherence set

$$A = [A] \ .$$

For such adherence the inside will be *diminishing* $A \supset \langle A \rangle$ and isotonic

$$A \subset B \Longrightarrow \langle A \rangle \subset \langle B \rangle \ .$$

The isotonic diminishing inside will be called *interior*. The set is open if it coincides with it's interior

$$A = \langle A \rangle \ .$$

3.2 We shall look the continuity properties for the reforms of adherence or inside spaces. We shall content himself with the case of mappings $f : X \longrightarrow Y$. Then the mapping of partial sets is provided by direct image $f^P : X^P \longrightarrow Y^P$. We shall use the predicates from the first and second octets only. Such predicates

are good for the sequential adherences, but we haven't opportunity to speak there about this subject now. The adherence operator and the adherence filters helps us to deal continuity properties more easy.

At first we shall check the equivalent predicates which coincide with property of traditional continuous mapping.

3.2.1 The respecting (-13) mapping of adherence spaces is defined by directed squaree:

$$\begin{tikzcd} \odot \arrow[r, "f^P"] & \cdot \\ \cdot \arrow[u, "\mathtt{Adh}_X"] \arrow[r, "f"'] \arrow[ur, phantom, "(-13)"] & \cdot \arrow[u, "\mathtt{Adh}_Y"'] \end{tikzcd} \nearrow$$

The corresponding inclusion of reforms

$$A \circ \mathtt{Adh}_X \circ f \subset A \circ f^P \circ \mathtt{Adh}_Y$$

is easily defined with adherence operators i.e. we get usual condition for continuous mappings in topological spaces

$$f^P([A]) \subset [f^P(A)] \uparrow A \subset X \ .$$

The reversely respecting $(+31)$ mapping is easily defined with adherence filters:

$$\nearrow \begin{tikzcd} \cdot & \cdot \arrow[l, "f^{P*}"'] \\ \cdot \arrow[u, "\mathtt{Adh}^*_X"] \arrow[ur, phantom, "(+31)"] & \odot \arrow[l, "f^*"] \arrow[u, "\mathtt{Adh}^*_Y"'] \end{tikzcd}$$

The inclusion of reforms

$$y \circ f^* \circ \mathtt{Adh}^*_X \subset y \circ \mathtt{Adh}^*_Y \circ f^{P*}$$

can be written as properties of adherence filters

$$\mathcal{F}_{f^{-1}(y)} \leq (f^P)^{-1}(\mathcal{F}_y) \ .$$

For every point in a target space $y \in Y$ every point of inverse image $x \in f^{-1}(y)$ is adherent only for such set $A \subset X$ which has direct image $f^P(A) \subset Y$ for which taken point is adherent $y \in [f^P(A)]$.

3.2.2 The reversely creating (-24) mapping of adherence spaces

$$\begin{tikzcd} \cdot & \odot \arrow[l, "f^{P*}"'] \\ \cdot \arrow[u, "\mathtt{Adh}_X"] \arrow[ur, phantom, "(-24)"] & \cdot \arrow[l, "f^*"] \arrow[u, "\mathtt{Adh}_Y"'] \end{tikzcd}$$

$\searrow$

is easily defined with adherence operators. The inclusion of reforms

$$B \circ f^{P*} \circ \mathtt{Adh}_X \subset B \circ \mathtt{Adh}_Y \circ f^*$$

we want to change with inclusion of adherence sets. For every partial set in the source space $A \subset X$ having the direct image coinciding with taken partial set in the target space $B = f^P(A)$, the adherence set is smaller than inverse image of taken partial set

$$A \subset f^{-1}(B) \ .$$

For isotonic adherences this property can be get more easily.

Proposition. *For reversely creating mapping $f : X \longrightarrow Y$ every covered partial set $B \subset f^P(X)$ has inclusion for adherence set of inverse image*

$$[f^{-1}(B) \subset f^{-1}([B]) \ .$$

If the adherence in the source space is isotone, then this property is also sufficient for mapping to be reversely creating.

If the adherence in the target space is isotone, then this property is equivalent to the same requirement for every partial set in the target space $B \subset Y$.

Proof: For covered set $B \subset f^P(X)$ the inverse image $A = f^{-1}(B)$ will have the image coinciding with the taken set $f^P(A) = B$, therefore its adherence set is included in the inverse image of adherence set for the taken set

$$[f^{-1}(B)] \subset f^{-1}([B]) \ .$$

For every partial set in the target space $B \subset Y$ we can take smaller set $B' = f^P(f^{-1}(B))$, and for isotone adherence in the target space

$$[f^{-1}(B)] = [f^{-1}(B')] \subset f^{-1}([B']) \subset f^{-1}([B]) \ .$$

For every partial set $A \subset X$ with image $f^P(A) = B$ we have $A \subset f^{-1}(A)$, therefore for isotone adherence in the source space we shall have

$$[A] \subset [f^{-1}(B)] \subset f^{-1}([B]) \ .$$

□

For isotonic adherences this property is equivalent with the inclusion of adherence for the inverse image

$$[f^{-1}(B)] \subset f^{-1}([B]) \ .$$

The creating (+42) mapping

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "f^P"'] \\ \odot \arrow[u, "\mathtt{Adh}^*_X"] \arrow[ur, phantom, "(+42)"] & \cdot \arrow[l, "f"] \arrow[u, "\mathtt{Adh}^*_Y"'] \end{tikzcd}$$

is defined by inclusion of reforms

$$x \circ \mathtt{Adh}^*_X \circ f^P \subset x \circ f \circ \mathtt{Adh}^*_Y \ .$$

This can be written with inclusion of image for adherence filter

$$f^{PP}(\mathcal{F}_x) \leq \mathcal{F}_{f(x)} \ .$$

Every partial set $A \subset X$ adherent by the taken point $x \in X$ has image $f^P(A) \subset Y$ adherent by the image of the taken point $y = f(x)$.

3.2.3 The pressing (-23) mapping

$$\begin{tikzcd} \cdot \arrow[d, "\mathtt{Adh}_X"'] \arrow[dr, phantom, "(-23)"] & \odot \arrow[l, "f^{P*}"'] \arrow[d, "\mathtt{Adh}_Y"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

is defined by inclusion of reforms

$$B \circ f^{P*} \circ \mathtt{Adh}_X \circ f \subset B \circ \mathtt{Adh}_Y \ .$$

It can be expressed as small image of adherence $\mathtt{Adh}_X \subset X^P \times X$

$$(f^P \times f)^P(\mathtt{Adh}_X) \subset \mathtt{Adh}_Y \ .$$

For the adherence operator we shall get small images

$$f^P([A]) \subset [B]$$

for the adherent set of sets $A \subset X$ which has direct image coinciding with taken set $f^P(A) = B$. This can be written as inclusion of the filters

$$f^{PP}([(f^P)^{-1}(B)]) \leq \{[B]\} \ .$$

For isotone adherences this property is expressed more easily.
Proposition. *For the isotone adherence in the source space the mapping is pressing if every covered partial set $B \subset Y$ has small image of adherence set for the inverse image of the taken set*

$$f^P([f^{-1}(B)]) \subset [B] \ .$$

For the isotone adherence in the target space this requirement is equivalent to such inclusion for every partial set $B \subset Y$.
Proof: For isotone adherence in the source space a smaller partial set will have a smaller adherence set

$$A \subset f^{-1}(B) \Longrightarrow [A] \subset [f^{-1}(B)] \ ,$$

therefore we shall have the inclusion for the image of adherence set

$$f^P([A]) \subset f^P([f^{-1}(B)]) \ .$$

We have shown that for such adherence every partial set $A \subset X$ will have a small image of adherence set

$$f^P([A]) \subset [B] \ .$$

If the adherence in the target space is isotone, then for every partial set $B \subset Y$ we have a smaller covered set $f^P(f^{-1}(B)) \subset B$, and we can check the needed inclusion

$$f^P([f^{-1}(B)]) \subset [f^P(f^{-1}(B))] \subset [B] \ .$$

□

We have shown that for isotone adherences the property of pressing mapping is expressed with inverse image of taken set in the target space

$$f^P([f^{-1}(B)]) \subset [B] \uparrow B \subset Y \ .$$

The reversely pressing (+32) mapping

$$\begin{tikzcd}
\cdot \arrow[r, "f^P"] & \cdot \\
\cdot \arrow[u, "\mathtt{Adh}^*_X"] \arrow[ur, phantom, "(+32)"] & \odot \arrow[l, "f^*"] \arrow[u, "\mathtt{Adh}^*_Y"']
\end{tikzcd}$$

is defined by inclusion of opposite reforms

$$y \circ f^* \circ \mathtt{Adh}^*_X \circ f^P \subset y \circ \mathtt{Adh}^*_Y \ ,$$

which also express small image of adherence

$$(f \times f^P)^P(\mathtt{Adh}^*_X) \leq \mathtt{Adh}^*_Y \ .$$

For adherence filters we get that for any point in a target space $y \in Y$ the points of inverse image $x \in f^{-1}(y)$ have adherence filter $\mathcal{F}_x$ with small image

$$f^{PP}(\mathcal{F}_{f^{-1}(y)}) \leq \mathcal{F}_y \ .$$

3.2.4 The lavishing (-14) mapping

$$\begin{tikzcd}
 & \odot \arrow[d, "\mathtt{Adh}_X"'] \arrow[dr, phantom, "(-14)"] & \cdot \arrow[l, "f^P"'] \arrow[d, "\mathtt{Adh}_Y"] \\
 & \cdot & \cdot \arrow[l, "f^*"] \\
\searrow & &
\end{tikzcd}$$

demands an inclusion of reforms

$$A \circ \mathtt{Adh}_X \subset A \circ f^P \circ \mathtt{Adh}_Y \circ f^* \ ,$$

i.e. we ask a large inverce image of the adherence

$$\mathtt{Adh}_X \subset (f^P \times f)^{-1}(\mathtt{Adh}_Y) \ .$$

For adherence operator it will be an inclusion of adherence set

$$[A] \subset f^{-1}([f^P(A)]) \uparrow A \subset X \ .$$

The reversely lavishing $(+41)$ mapping

$$\begin{tikzcd}
\nearrow & & \\
 & \cdot \arrow[dr, phantom, "(+41)"] & \cdot \arrow[l, "f^{P*}"'] \\
 & \odot \arrow[u, "\mathtt{Adh}^*_X"] \arrow[r, "f"'] & \cdot \arrow[u, "\mathtt{Adh}^*_Y"']
\end{tikzcd}$$

asks another inclusion

$$\mathtt{Adh}^*_X \subset f \circ \mathtt{Adh}^*_Y \circ f^{P*} \ ,$$

which again is equivalent of asking the large inverse image of adherence

$$\mathtt{Adh}^*_X \subset (f \times f^P)^{-1}(\mathtt{Adh}^*_Y) \ .$$

It can be expressed with large inverse image of adherence filter

$$\mathcal{F}_x \leq (f^P)^{-1}(\mathcal{F}_{f(x)}) \ .$$

3.3 Remain the predicates without equivalent counterparts.

3.3.1 Wasting $(+13)$ mapping

$$\begin{tikzcd}
\odot \arrow[r, "f^P"] \arrow[d, "\mathtt{Adh}_X"'] \arrow[dr, phantom, "(+13)"] & \cdot \arrow[d, "\mathtt{Adh}_Y"] & \\
\cdot \arrow[r, "f"'] & \cdot & \swarrow
\end{tikzcd}$$

asks an inclusion of reforms

$$A \circ f^P \circ \mathtt{Adh}_Y \subset A \circ \mathtt{Adh}_X \circ f \ .$$

It can be expressed with adherence operators

$$[f^P(A)] \subset f^P([A]) \uparrow A \subset X \ .$$

The reversely wasting (-31) mapping

$$\begin{tikzcd}
\swarrow & & \\
 & \cdot \arrow[dr, phantom, "(-31)"] & \cdot \arrow[l, "f^{P*}"'] \\
 & \cdot \arrow[u, "\mathtt{Adh}^*_X"] & \odot \arrow[l, "f^*"] \arrow[u, "\mathtt{Adh}^*_Y"']
\end{tikzcd}$$

asks an inclusion of reforms

$$y \circ \mathtt{Adh}^*_Y \circ f^{P*} \subset y \circ f^* \circ \mathtt{Adh}^*_X \ .$$

This can be expressed with small inverse image of adherence filter

$$(f^P)^{-1}(\mathcal{F}_y) \subset \mathcal{F}_{f^{-1}(y)} \ .$$

3.3.2 The reversely cautious (+24) mapping

$$\begin{tikzcd}
& \cdot \arrow[d, "\mathtt{Adh}_X"'] \arrow[dr, phantom, "(+24)"] & \odot \arrow[l, "f^{P*}"'] \arrow[d, "\mathtt{Adh}_Y"] \\
\searrow & \cdot & \cdot \arrow[l, "f^*"]
\end{tikzcd}$$

is asking the inclusion of reforms

$$B \circ \mathtt{Adh}_Y \circ f^* \subset B \circ f^{P*} \circ \mathtt{Adh}_X \ .$$

This property can be expressed with adherence operator: Every point $x \in f^{-1}([B])$ in the inverse image of the adherence set for taken set in the target space $B \subset Y$ is adherent $x \in [A]$ for some partial set in the source space $A \subset X$ with image coinsiding with taken set $f^P(A) = B$.

Proposition. *For the isotone adherence in a source space the mapping $f : X \longrightarrow Y$ will be reversely cautious exactly then, when for every covered partial set $B \subset f^P(X)$ the inverse image of the adherence set would contained in the adherence set of the inverse image of taken set*

$$f^{-1}([B]) \subset [f^{-1}(B)] \ ,$$

and for other partial sets in a target space $B \not\subset f^P(X)$ the adherence set would contained in completment of mapping's whole image

$$[B] \subset Y \setminus f^P(X) \ .$$

Proof: This condition is obviously sufficient.

And it is necessary: If a point $x \in X$ is in inverse image of the adherence set $x \in f^{-1}([B])$ for a covered set $B \subset f^P(X)$, then any set $A \subset X$ with the image $f^P(A) = B$ and adherent by the taken point $x \in [A]$ will have $A \subset f^{-1}(B)$, therefore for isotone adherence

$$x \in [A] \subset [f^{-1}(B)] \ .$$

For other partial sets $B \not\subset f^P(X)$ the inverse image of the adherence set will be empty $f^{-1}([B]) = \emptyset$, as we can't find any partial set $A \subset X$ having image equal to the taken set, i.e.

$$f^P(A) \neq B \uparrow A \subset X \ .$$

□

The cautious (−42) mapping

$$\begin{tikzcd}
\cdot \arrow[dr, phantom, "(-42)"] & \cdot \arrow[l, "f^P"'] & \nwarrow \\
\odot \arrow[u, "\mathtt{Adh}^*_X"] \arrow[r, "f"'] & \cdot \arrow[u, "\mathtt{Adh}^*_Y"'] &
\end{tikzcd}$$

asks an inclusion of reforms

$$x \circ f \circ \mathtt{Adh}^*_Y \subset x \circ \mathtt{Adh}^*_X \circ f^P \ .$$

This is expressed with big image of the adherence filter

$$f^{PP}(\mathcal{F}_x) \geq \mathcal{F}_{f(x)} \ .$$

It is very special property for the mapping: the point's image $f(x) \in Y$ can be adherent only such partial sets $B \subset Y$ which coincide with direct image of any partial set in the source space $B = f^P(A)$.

3.3.3 The hiding (+23) mapping

$$
\begin{tikzcd}
\cdot \arrow[d, "\mathtt{Adh}_X"'] \arrow[dr, phantom, "(+23)"] & \odot \arrow[l, "f^{P*}"'] \arrow[d, "\mathtt{Adh}_Y"] \\
\cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}
$$

demands an inclusion of relations

$$B \circ \mathtt{Adh}_Y \subset B \circ f^{P*} \circ \mathtt{Adh}_X \circ f \ ,$$

which equivalent to the large image of adherence

$$\mathtt{Adh}_Y \subset (f^P \times f)^P(\mathtt{Adh}_X) \ .$$

This can be expressed with adherence operator. For the adherent point $y \in [B]$ we can find partial set in a source space $A \subset X$ which image coincides with the taken set $f^P(A) = B$ and we have adherent point $x \in [A]$ covering the earlier point in the target space $f(x) = y$

$$[B] \subset \bigcup\{f^P([A]) : f^P(A) = B\} \ .$$

For the isotone adherence in the source space the hiding mapping can be defined by stronger requirement.
Proposition. *If the whole target set has any adherent point $[Y] \neq \emptyset$, then hiding mapping $f : X \longrightarrow Y$ will cover the whole target space.*

For the isotone adherence in the source space $\mathtt{Adh}_X$ the hiding mapping $f : X \longrightarrow Y$ will provide every covered partial set $B \subset f^P(X)$ with adherence set contained in the image of adherence set for inverse image of taken set

$$[B] = \emptyset \uparrow B \not\subset f^P(X) \ .$$

Proof: For not covered set $B \subset Y$ we cant find any partial set in the source space $A \subset X$ with coinciding image $f^P(A) = B$, therefore for hiding mapping such set cannot have any adherent point $[B] = \emptyset$.

For the covered set $B \subset Y$ every adherent point $y \in [B]$ must be covered by adherent point $x \in [A]$ for some partial set in the source space $A \subset X$ with coinciding image $f^P(A) = B$. If the adherence in the source space is isotone, then

$$y \in f^P([A]) \subset f^P([f^{-1}(B)]) \uparrow B \subset Y \ .$$

□

The reversely hiding (−32) mapping

$$
\begin{tikzcd}
\cdot \arrow[r, "f^P"] \arrow[dr, phantom, "(-32)"] & \cdot \\
\cdot \arrow[u, "\mathtt{Adh}^*_X"] & \odot \arrow[l, "f^*"] \arrow[u, "\mathtt{Adh}^*_Y"']
\end{tikzcd}
$$

demands an inclusion of reforms

$$y \circ \mathtt{Adh}^*_Y \subset y \circ f^* \circ \mathtt{Adh}^*_X \circ f^P \ ,$$

which is equivalent for large image of adherence's opposite reform

$$\mathtt{Adh}^*_Y \leq (f \times f^P)^P(\mathtt{Adh}^*_X) \ .$$

This will be expressed as large image of adherence filter appointment

$$\mathcal{F}_y \leq \bigcup\{f^{PP}(\mathcal{F}_x) : x \in f^{-1}(x)\} \ .$$

We have seen that such mapping must be very special. For the target space with adherent point $[Y] \neq \emptyset$ such mapping must be covering.

3.3.4 The reflecting (+14) mapping

$$\begin{tikzcd} \odot \arrow[r, "f^P"] \arrow[d, "\mathtt{Adh}_X"'] \arrow[dr, phantom, "(+14)"] & \cdot \arrow[d, "\mathtt{Adh}_Y"] \\ \cdot & \cdot \arrow[l, "f^*"] \end{tikzcd}$$

is defined by inclusion of reforms

$$A \circ f^P \circ \mathtt{Adh}_Y \circ f^* \subset A \circ \mathtt{Adh}_X \ ,$$

which is equivalent to the small inverse image of the adherence's graphic

$$(f^P \times f)^{-1}(\mathtt{Adh}_Y) \subset \mathtt{Adh}_X \ .$$

This can be expressed with the adherence operator as large adherence set in the source space

$$f^{-1}([f^P(A)]) \subset [A] \uparrow A \subset X \ .$$

The adherent point of some set $x \in [A]$ is a property which is reflected by such mapping. If this property is checked in the target space for image point and the set $f(x) \in [f^P(A)]$, then we can conclude that this property is also content for initial point and partial set in the source space.

The reversely reflecting (-41) mapping

$$\begin{tikzcd} \cdot \arrow[dr, phantom, "(-41)"] & \cdot \arrow[l, "f^{P*}"'] \\ \odot \arrow[u, "\mathtt{Adh}^*_X"] \arrow[r, "f"'] & \cdot \arrow[u, "\mathtt{Adh}^*_Y"'] \end{tikzcd}$$

is defined by inclusion of reforms

$$x \circ f \circ \mathtt{Adh}^*_Y \circ f^{P+} \subset x \circ \mathtt{Adh}^*_X \ ,$$

which is equivalent to the *small inverse image* of adherence's opposite reform

$$(f \times f^P)^{-1}(\mathtt{Adh}^*_Y) \leq \mathtt{Adh}^*_X \ .$$

This property can be expressed for the appointment of adherence filters

$$(f^P)^{-1}(\mathcal{F}_{f(x)}) \leq \mathcal{F}_x \ .$$

3.4 Finely we formulate the predicates from the third octet. These properties is not useful for the traditional topological adherences. But they may be interesting for some integral representation of set's middle point. The adherence point $x \in [X]$ can be understood as middle point of this set. Such point usually exists and is defined uniquely, but we can questionize such existence or may exist many of needed middle points for some extraordinary sets.

3.4.1 The binding (+12) mapping

$$\begin{tikzcd} \odot \arrow[r, "f^P"] \arrow[d, "\mathtt{Adh}_X"'] \arrow[dr, phantom, "(+12)"] & \cdot \\ \cdot \arrow[r, "f"'] & \cdot \arrow[u, "\mathtt{Adh}^*_Y"'] \end{tikzcd}$$

is defined by inclusion of reforms

$$A \circ f^P \subset A \circ \mathtt{Adh}_X \circ f \circ \mathtt{Adh}^*_Y \ ,$$

which is equivalent to the large inverse image of the taken mapping

$$f^P \subset (\mathtt{Adh}_X \times Adj_Y)^{-1}(f) \ .$$

It may be defined with the union of adherence filters

$$f^P(A) \in \mathcal{F}_{f^P([A])} \ ,$$

the image of the partial set $A \subset X$ has the image $f(y) \in Y$ of the middle point $x \in [A]$ as a middle point.

For functional convergence in the source space this property would be the same as for the creating mapping (+42), and for functional convergence in the target space this property would be the same as for the wasting mapping (+13), therefore it is a same as usual continuity property for mappings defined with reversely respecting (+31), reversely pressing (+32), reversely lavishing (+14) mappings.

The reversely binding (−21) mapping

$$\begin{tikzcd}
\swarrow & \cdot & \odot \arrow[l, "f^{P*}"'] \arrow[d, "\mathtt{Adh}_Y"] \\
& \cdot \arrow[u, "\mathtt{Adh}_X^*"] \arrow[ur, phantom, "(-21)"] & \cdot \arrow[l, "f^*"]
\end{tikzcd}$$

is defined by inclusion of reforms

$$B \circ f^{P*} \subset B \circ \mathtt{Adh}_Y \circ f^* \circ \mathtt{Adh}_X \ ,$$

which is equivalent to the large inverse image of the inverse napping

$$f^{P*} \subset (\mathtt{Adh}_Y \times \mathtt{Adh}_X)^{-1}(f^*) \ .$$

This property can also be expressed with union of adherence filters

$$(f^P)^{-1}(B) \leq \mathcal{F}_{f^{-1}([B])} \ ,$$

every partial set $A \subset X$ which has the image $f^P(A) = B$ coinciding with the taken partial set in the target space $B \subset Y$, has some adherent point in the inverse image $x \in f^{-1}(B)$ of taken set $B \subset Y$.

3.4.2 The thick (+43) mapping

$$\begin{tikzcd}
\cdot \arrow[r, "f^P"] & \cdot \arrow[d, "\mathtt{Adh}_Y"] & \\
\odot \arrow[u, "\mathtt{Adh}_X^*"] \arrow[r, "f"'] \arrow[ur, phantom, "(+43)"] & \cdot & \swarrow
\end{tikzcd}$$

is defined by an inclusion of reforms

$$x \circ \mathtt{Adh}_X^* \circ f^P \circ \mathtt{Adh}_Y \subset x \circ f \ ,$$

which is equivalent for small image of mapping's direct image appointment

$$(\mathtt{Adh}_X \times \mathtt{Adh}_Y)^P(f^P) \subset f \ .$$

This can be expressed with adherence operator's union over all sets from some filter

$$\bigcup\{[B] : B \in f^{PP}(\mathcal{F}_x)\} \subset \{f(x)\} \ ,$$

if the taken point $x \in X$ adheres the partial set $A \subset X$, then the image point $f(x) \in Y$ is a unique adherent point for the image set $f^P(A) \subset Y$.

For the functional convergence $\mathtt{Adh}_X$ in the source space we get the equivalent property of wasting mapping (+13). For the functional convergence $\mathtt{Adh}_Y$ in the target space we get the equivalent property of

creating mapping (+42), therefore for mappings this is equivalent with reversely respecting (+31), reversely pressing (+32), reversely lavishing (+41) mappings also.

3.5 Now rest more special properties, without any equivalent counterparts.

3.5.1 The thin (−12) mapping

$$\begin{tikzcd}
\odot \arrow[r, "f^P"] \arrow[d, "\mathtt{Adh}_X"'] \arrow[dr, phantom, "(-12)"] & \cdot & \nwarrow \\
\cdot \arrow[r, "f"'] & \cdot \arrow[u, "\mathtt{Adh}_Y^*"'] &
\end{tikzcd}$$

is defined by an inclusion of reforms

$$A \circ \mathtt{Adh}_X \circ f \circ \mathtt{Adh}_Y^* \subset A \circ f^P \ ,$$

which is eqiuvalent to the small inverse image of mapping

$$(\mathtt{Adh}_X \times \mathtt{Adh}_Y)^{*P}(f) \subset f^P \ .$$

This can be expressed with the union of adherence filters

$$\mathcal{F}_{f^P([A])} \subset \{f^P(A)\} \ ,$$

the image $f^P(A) \subset Y$ of every partial set $A \subset X$ in the source space is unique partial set $B \subset Y$ in the target space which has an adherent point $y \in [B]$ related to some adherent point of the taken partial set $x \in [A]$, $f(x) = y$. This means that the related adherence points must also relate the partial sets.

The reversely thin (+21) mapping

$$\begin{tikzcd}
\nearrow & \cdot \arrow[d, phantom, ""] & \odot \arrow[l, "f^{P*}"'] \arrow[d, "\mathtt{Adh}_y"] \\
& \cdot \arrow[u, "\mathtt{Adh}_X^*"] \arrow[ur, phantom, "(+21)"] & \cdot \arrow[l, "f^*"]
\end{tikzcd}$$

is defined by inclusion of reforms

$$B \circ \mathtt{Adh}_Y \circ f^* \circ \mathtt{Adh}_X^* \subset B \circ f^{P*} \ ,$$

which is equivalent to the small inverse image of inverse mapping's graphic

$$(\mathtt{Adh}_Y \times \mathtt{Adh}_Y)^{*P}(f^*) \subset f^{P*} \ .$$

This can be expressed with the union of adherence filters

$$\mathcal{F}_{f^{-1}([B])} \subset (f^P)^{-1}(B) \ ,$$

the partial set $A \subset X$, with the adherent points $x \in [A]$ having related points $y \in Y$, $f(x) = y$ adherent to the taken partial set $y \in [B]$, must be related to the taken set in the target space $f^P(A) = B$.

3.5.2 The parting (−43) mapping

$$\begin{tikzcd}
\cdot \arrow[r, "f^P"] \arrow[dr, phantom, "(-43)"] & \cdot \arrow[d, "\mathtt{Adh}_Y"] & \\
\odot \arrow[u, "\mathtt{Adh}_X^*"] \arrow[r, "f"'] & \cdot & \nearrow
\end{tikzcd}$$

is defined by inclusion of reforms

$$x \circ f \subset x \circ \mathtt{Adh}_X \circ f^P \circ \mathtt{Adh}_Y \ ,$$

which is equivalent to a large image of the mapping's direct image appointment

$$f \subset (\mathtt{Adh}_X \times \mathtt{Adh}_Y)^P(f^P) \ .$$

The adherence expands set image so much that it include the mapping itself $f : X \longrightarrow Y$. This can be expressed with upper covering of adherence operator over the adherence filter

$$\{f(x)\} \subset \bigcup\{[B] : B \in f^{PP}(\mathcal{F}_x)\} \ ,$$

i. e. the image $f(x) \in Y$ of a taken point $x \in X$ is adherent to the image $f^P(A) \subset Y$ of any partial set in the source space $A \subset X$ for which the taken point is adherent $x \in [A]$.

If this adherent point is defined as set's middle point

$$x = \int A \ ,$$

then the parting mapping asks that the image $f(x) \in Y$ would be a middle point of an direct image

$$f(x) = \int f^P(A)$$

for some set $A \subset X$ for which taken point is middle point.

This property is very week. For expanding adherences such property has every mapping $f : X \longrightarrow Y$. We can take $A = \{x\}$, then $x \in [A]$ and $f(x) \in [A]$.

The reversely parting (+34) mapping

$$\begin{tikzcd}
\cdot \arrow[d, "\mathtt{Adh}_X"'] \arrow[dr, phantom, "(+34)"] & \cdot \arrow[l, "f^{P*}"'] \\
\cdot & \odot \arrow[l, "f^*"] \arrow[u, "\mathtt{Adh}_Y^*"']
\end{tikzcd}$$

is defined by inclusion of reforms

$$y \circ f^* \subset y \circ \mathtt{Adh}_Y^* \circ f^{P*} \circ \mathtt{Adh}_X \ ,$$

which is equivalent to large image of inverse of mapping's direct image appointment

$$f^* \subset (\mathtt{Adh}_Y \times \mathtt{Adh}_X)^P(f^{P*}) \ .$$

This can be expressed with upper covering of adherence operator over the filter

$$f^{-1}(y) \subset \bigcup\{[A] : f^P(A) \in \mathcal{F}_y\} \ ,$$

i.e. for the point in the target space $y \in Y$ every point of inverce image $x \in f^{-1}(y)$ is adherent some partial set $A \subset X$ which direct image $f^P(A) \subset Y$ is adherent by the taken point $y \in [f^P(A)]$.

3.6 We can define weakly cautious mappings of adherence spaces, and to show that such property is equivalent to respecting mappings of inside spaces. This provides the fact, that some continuity properties may be expressed both in adherence and inside spaces. More details is presented in my recent monograph [13].

4. Mapping's global continuity

4.1 The continuity properties in adherence spaces provide local continuity properties in generated topological spaces. The same situation also arise for bornological or measurable spaces. Therefore it is usefull to look at general continuity properties for topological spaces. At this moment we content himeself to deal only with mappings. These continuity properties will be called global.

4.2 The part in the potential set $\mathcal{T} \subset X^P$ will be called a *tribe* and it will define a *tribe space* X. For any mapping $f : X \to Y$ with set's direct image appointment $f^P : X^P \to Y^P$ we can define 3 continuity properties: a *carrying mapping* $f^{PP}(\mathcal{T}_X) \leq \mathcal{T}_Y$, a *hiding mapping* $f^{PP}(\mathcal{T}_X) \geq \mathcal{T}_Y$, and an *image reflecting mapping* $(f^P)^{-1}(\mathcal{T}_Y) \leq \mathcal{T}_X$.

For topological spaces we can investigate the topology $\mathcal{T}$ and cotopology $\mathcal{T}^c$ together, therefore we get different continuity properties: For topology compounded by closed sets a *closed mapping* $f^{PP}(\mathcal{T}_X) \leq \mathcal{T}_Y$, a *closed set hiding mapping* $f^{PP}(\mathcal{T}_X) \geq \mathcal{T}_Y$, and an *closed sets image reflecting mapping* $(f^P)^{-1}(\mathcal{T}_Y) \leq \mathcal{T}_X$, and for cotopology compounded by open sets a *open mapping* $f^{PP}(\mathcal{T}_X^c) \leq \mathcal{T}_Y^c$, a *open sets hiding mapping* $f^{PP}(\mathcal{T}_X^c) \geq \mathcal{T}_Y^c$, and an *open set image reflecting mapping* $(f^P)^{-1}(\mathcal{T}_Y^c) \leq \mathcal{T}_X^c$.

With set's inverse image appointment $f^{-1} : Y^P \to X^P$ we define another 3 continuity properties: a *turning mapping* $(f^{-1})^P(\mathcal{T}_Y) \leq \mathcal{T}_X$, a *exhausting mapping* $(f^{-1})^P(\mathcal{T}_Y) \geq \mathcal{T}_X$, and an *inverse image reflecting mapping* $(f^{-1})^{-1}(\mathcal{T}_Y) \leq \mathcal{T}_X$.

For topological spaces only the turning mappings are concidered as continuous ones.

4.2.1 We defined the closure as isotonic and expanding adherence. The closed sets is defined as fixed points for closure operator, i.e. the closed sets are defined by the property

$$A \subset [A] \subset A \ .$$

The tribe compounded by all closed sets is called a topology. It is characterized by the property of freely intersection

$$\bigcap \mathcal{E} \in \mathcal{T} \uparrow \mathcal{E} \subset \mathcal{T} \ ,$$

every tribe, which maintains the arbitrary intersection of its members, can be get as topology of closed sets for some closure. Such topologies are used in logic calculations [12]. Bourbaki for more traditional topologies asks additional property that wide set is closed.

Otherwise the closure was called a multistep (germ. mehrstufig) topology by Gähler [7].

The respecting (-13) mapping asks inclusion for direct image

$$f^P([A]) \subset [f^P(A)] \uparrow A \subset X \ ,$$

and reversely creating mapping (-24) asks equivalent (for mappings of closure spaces) opposite inclusion for inverse image

$$[f^{-1}(B)] \subset f^{-1}([B]) \uparrow B \subset Y \ .$$

This can be also expressed by equivalent (for mappings) properties with direct or inverse images of closure reform. The pressing (-23) mapping asks an inclusion of direct image

$$(f^P \times f)^P(\mathtt{Adh}_X) \subset Adh_Y \ ,$$

and lavishing (-14) mapping asks the same opposite inclusion of inverse image

$$\mathtt{Adh}_X \subset (f^P \times f)^{-1}(\mathtt{Adh}_Y) \ .$$

These all equivalent continuity properties coincide with usual continuity of mappings between closure spaces.

Such mapping is turning for topological spaces, i.e. the inverse image of closed set remains closed

$$[B] \subset B \Longrightarrow [f^{-1}(B)] \subset f^{-1}([B]) \subset f^{-1}(B) \ .$$

The wasting (-31) mapping asks opposite inclusion for direct image

$$[f^P(A)] \subset f^P([A]) \uparrow A \subset X \ .$$

It defines the closed mapping between closure spaces: The direct image of closed set remains closed

$$[A] \subset A \Longrightarrow [f^P(A)] \subset f^P([A]) \subset f^P(A) \ .$$

Such mappings are carrying for topological spaces.

Bourbaki [4] concidered the property of closed mappings only together with continuity.

The cautious (-42) mappings asks big image of the adherence filter

$$f^{PP}(\mathcal{F}_x) \geq \mathcal{F}_{f(x)} \ .$$

For covering mapping between closure spaces it asks an inclusion of inverse image

$$f^{-1}([B]) \subset [f^{-1}(B)] \uparrow B \subset Y \ .$$

The more useful property of *weakly cautious mapping*. It demands that image of adherence filter would generate a big hereditary filter

$$[f^{PP}(\mathcal{F}_x)] \geq \mathcal{F}_{f(x)} \ .$$

Such mapping will be equivalent to respecting mapping of conjugate interior spaces. It will be open for generated topology.

Let $0 \subset X$ is open set of closure in a source space. It is easy to check that the direct image $f^P(O) \subset Y$ will be open set of closure in a target space. It is enough to show that any set $A' \subset Y \setminus f^P(O)$ has no adherent point from image of taken set $y \in f^P(O)$. Otherwise we should have a point $x \in O$ with $f(x) = y$, and for weakly cautious mapping this point will be an adhered by the set $A \subset X$ with smaller direct image $f^P(A) \subset A'$. Therefore $A \subset X \setminus O$, and such set can't adhere the point from the taken open set $x \in O$. So we get a contradiction, and there is no point from the taken open set adhered by the set $A' \subset Y \setminus f^P(O)$.

The hiding (-32) mapping asks the opposite inclusion of direct image of closure reform

$$(f^P \times f)^P(\mathtt{Adh}_X) \supset \mathtt{Adh}_Y \ ,$$

and reflecting $(+14)$ mapping asks the inclusion of inverse image of closure reform

$$(f^P \times f)^{-1}(\mathtt{Adh}_Y) \subset \mathtt{Adh}_X \ .$$

Such local properties don't coincide with corresponding global properties in topological spaces.

4.2.2 We define the interior as isotonic and decreasing adherence. The open sets is defined as fixed points for interior operator, i.e. the open sets are defined by the property

$$A \supset \langle A \rangle \supset A \ .$$

The tribe compounded by all open sets we shall call a cotopology. It is characterized by the property of freely union

$$\bigcup \mathcal{E} \in \mathcal{K} \uparrow \mathcal{E} \subset \mathcal{K} \ ,$$

every tribe, which maintains the arbitrary union of its members, can be get as cotopology of open sets for some interior. Cotopologies are dual for topologies, however the continuity properties of mappings are different.

From continuous mappings of closure sets with inverse image condition

$$[f^{-1}(B)] \subset f^{-1}([B]) \uparrow B \subset Y \ ,$$

we get the dual inverse image condition for mappings of interior spaces

$$f^{-1}(\langle B \rangle) \subset \langle f^{-1}(B) \rangle \uparrow B \subset Y \ .$$

Such mappings are turning between the cotopological spaces

$$B \subset \langle B \rangle \Longrightarrow f^{-1}(B) \subset f^{-1}(\langle B \rangle) \subset \langle f^{-1}(B) \rangle \ .$$

Nevertheless this condition cannot be get as earlier continuity property for adherence spaces. The cautious (-42) mappings between interior spaces is defined by the same inclusion for covered subsets

$$f^{-1}(\langle B \rangle) \subset \langle f^{-1}(B) \rangle \uparrow B \subset f^P(X) \ ,$$

and in the target space we demand that

$$\langle B\rangle \subset Y \setminus f^P(X) \uparrow B \not\subset f^P(X) \ .$$

Therefore in most cases the cautious mappings must be surjective.

The situation is rescued with the notion of *weakly cautious mapping* between interior spaces. This property is equivalent to the property of respecting mapping for conjugate closure spaces.

The respecting (-13) mapping between interior spaces is defined by inclusion of direct image

$$f^P(\langle A\rangle) \subset \langle f^P(A)\rangle \uparrow A \subset X \ .$$

It defines open mapping between topological spaces. Such mapping is carrying for cotopological spaces, i.e. the direct image of open set remains open

$$A \subset \langle A\rangle \Longrightarrow f^P(A) \subset f^P(\langle A\rangle) \subset \langle f^P(A)\rangle \ .$$

Bourbaki in [4] uses such notion together with continuity, ant doesn't remark any equivalent formulation of open mappings.

The reversely creating (-24) mapping between interior spaces is defined by opposite inclusion of inverse image

$$\langle f^{-1}(B)\rangle \subset f^{-1}(\langle B\rangle) \uparrow B \subset Y \ .$$

The pressing (-23) mapping between interior spaces demands the inclusion of direct image of interior reform

$$(f^P \times f)^P(\mathtt{Int}_X) \subset \mathtt{Int}_Y \ .$$

The lavishing (-14) mapping between interior spaces demands the opposite inclusion of inverse image of interior reform

$$\mathtt{Int}_X \subset (f^P \times f)^{-1}(\mathtt{Int}_Y) \ .$$

All such mappings define an open mapping of topological spaces.

The wasting (-31) mapping between interior spaces demands the opposite inclusion of direct image

$$f^P(\langle A\rangle) \supset \langle f^P(A)\rangle \uparrow A \subset X \ .$$

This property wasn't remarked for classical topological spaces, nevertheless it is meaningful for interesting topologies.

The hiding (-32) mapping between interior spaces demands the opposite inclusion of direct image of interior reform

$$(f^P \times f)^P(\mathtt{Int}_X) \supset \mathtt{Int}_Y \ .$$

The reflecting $(+14)$ mapping between interior spaces demands the the inclusion of inverse image of interior reform

$$(f^P \times f)^{-1}(\mathtt{Int}_Y) \subset \mathtt{Int}_X \ .$$

Both last local properties for mappings of cotopological spaces differ from coresponding global properties.

4.2.3 Any tribe $\mathcal{K} \subset X^P$ can be declared as bornology, and its member $B \subset X$ as bounded set. The carrying mapping between bornological spaces $f : X \longrightarrow Y$ is called bounded, i.e. the image of bounded set must be bounded

$$f^P(A) \in \mathcal{K}_Y \uparrow A \in \mathcal{K}_X \ .$$

The turning mapping between bornological spaces $f : X \longrightarrow Y$ we shall call perfect, i.e. the inverse image of bounded set must be bounded

$$f^{-1}(B) \in \mathcal{K}_X \uparrow B \in \mathcal{K}_Y \ .$$

Such mappings are useful for bornologies compounded of compact parts in some topological space.

4.2.3.1 It is interesting to consider adherence space as example of "local" bornology. The reform $\mathtt{Sid}_X : X^P \to\!\!\!- X$ will be called *proximity*. We say that the couple $\langle A, x\rangle \in X^P \times X$ belongs to proximity if the set

A is *proximal* to the point $x \in X$. The direct appointment for each set $A \subset X$ appoints the set of points for which taken set is proximal, we shall call it a *siding set* with sign $\langle A \rangle \subset X$. The opposite appointment for every point $x \in X$ appoints a *proximal filter* $\mathcal{F}_x$ of proximal sets

4.2.3.2 In proximity space we define bornology declaring all sets with biger siding set

$$A \subset \langle A \rangle \uparrow A \subset X$$

as bounded sets.

In bornology space we define proximity, declaring the set $A \subset X$ proximal to the point $x \in X$, if we can find a biger bounded set $A \subset B \subset X$ with taken point $x \in B$.

We shall check continuity properties for mappings between proximity spaces. Usually we work with proximal cofilters, demanding that smaller set remainded bounded and proximal.

4.2.3.3 The respecting (-13) mapping demands inclusion of siding sets

$$f^P(\langle A \rangle) \subset \langle f^P(A) \rangle \ .$$

Such mappings will be bounded for generated bornology spaces:

Let the set $A \subset X$ is near of every it's point

$$A \subset \langle A \rangle \ .$$

Then the direct image of taken set $f^P(A) \subset Y$ has the same property

$$f^P(A) \subset f^P(\langle A \rangle) \subset \langle f^P(A) \rangle \ .$$

And in opposite direction. The bounded mappings of bornological spaces are respecting for generated proximity spaces:

Let the set $A \subset X$ is near the point $x \in X$, i. e. we find a bigger bounded set $A \subset B$ having the taken point $x \in B$. We shall check that the direct image of taken set $f^P(A) \subset Y$ remains near the point $f(x) \in Y$. It is enough to take bounded set $f^P(B) \subset Y$, for which we have $f^P(A) \subset f^P(B)$ and $f(x) \in f^P(B)$.

The reversely respecting $(+31)$ mapping demands inclusion of proximal filters in source space

$$\mathcal{F}_{f^{-1}(y)} \leq (f^P)^{-1}(\mathcal{F}_y) \ .$$

It is equivalent for creating $(+42)$ mapping, which demands inclusion of proximal filters in target space

$$f^{PP}(\mathcal{F}_x) \leq \mathcal{F}_{f(x)} \ .$$

Therefore this property is similar to local boundedness.

There are additional two equivalent properties defined with direct or inverse images of the proximity reform.

The reversely pressing $(+32)$ mapping between proximity spaces demands a small direct image

$$(f^P \times f)^P(\mathtt{Sid}^*_X) \subset \mathtt{Sid}_Y \ ,$$

and reversely lavishing $(+41)$ mapping between proximity spaces demands a big inverse image

$$\mathtt{Sid}^*_X \subset (f^P \times f()^{-1}(\mathtt{Sid}^*_Y) \ .$$

4.2.3.4 Rest 4 more special continuity properties.

The cautious (-42) mapping between proximity spaces demands big direct image of proximity filter

$$f^{PP}(\mathcal{F}_x) \geq \mathcal{F}_{f(x)} \ .$$

We can show that such mappings are perfect for generated bornologies: The inverse image $f^{-1}(B') \subset X$ of bounded set $B' \subset Y$ remains bounded.

Let the set $B' \subset Y$ has a bigger siding set $B' \subset \langle B' \rangle$. We shall check that the same is also an inverse image

$$f^{-1}(B') \subset \langle f^{-1}(B') \rangle \ .$$

The smaller set $B'' \subset B'$ remains siding, therefore a smaller set remains bounded

$$B'' \subset B' \subset \langle B' \rangle \subset \langle B'' \rangle \ .$$

We can take a smaller bounded set $B'' := f^P(f^{-1}(B')) \subset B'$ covered by mapping, therefore for cautious mapping we get

$$f^{-1}(\langle B'' \rangle) \subset \langle f^{-1}(B'') \rangle \ ,$$

$$f^{-1}(B') = f^{-1}(B'') \subset f^{-1}(\langle B'' \rangle \subset \langle f^{-1}(B'') \rangle = \langle f^{-1}(B') \rangle \ .$$

And in opposite direction. For perfect covering mapping we get reversely cautious mapping (+24) of generated proximity spaces, i. e. for the set $A' \subset Y$ the inverse image of siding set $f^{-1}(\langle A' \rangle) \subset X$ must be included in union of siding sets

$$f^{-1}(\langle A' \rangle) \subset \bigcup \{ \langle A \rangle : f^P(A) = A' \} \ .$$

Every siding point $y \in \langle A' \rangle$ is included in bigger bounded set $y \in B'$, $A' \subset B'$. We can take the inverse image of this bounded set $f^{-1}(y) \subset f^{-1}(B')$ and it will be bigger for any set $A \subset X$ with $f^P(A) = A'$. For covering mapping we can take $A := f^{-1}(A')$, therefore

$$f^{-1}(\langle A' \rangle) \subset \langle f^{-1}(A') \rangle \ .$$

The reflecting mappings (+14) demmands small inverse image of siding reform

$$(f^P \times f)^{-1}(\mathtt{Sid}_Y) \subset \mathtt{Sid}_X \ .$$

We shall check that it reflects bounded sets for generated bornologies.

Let we have the set $A \subset X$ with bounded direct image

$$f^P(A) \subset \langle f^P(A) \rangle \ .$$

We shall show that taken set is also bounded

$$A \subset \langle A \rangle \ .$$

Every point $x \in A$ has image $f(x) \in Y$ proximal to the direct image of taken set $f^P(A) \subset Y$, therefore for reflecting mapping it will be proximal to the taken set $x \in \langle A \rangle$.

And in oposite direction. The mapping $f : X \to Y$ which reflects bounded sets will be also reflecting for generated proximities.

Let $f(x) \in Y$ is siding to the set $f^P(A) \subset Y$, i. e. we can find a bigger bounded set $f^P(A) \subset B'$ with $f(x) \in B'$. We can take $B := f^{-1}(B')$, it's image remains bounded $f^P(A) \subset f^P(B) \subset B'$, therefore for reflecting mapping it itself is bounded. We have find a bigger bounded set $A \subset B$ with $x \in B$.

The hiding (+23) mapping between siding spaces demands large direct image of siding reform

$$\mathtt{Sid}_Y \subset (f^P \times f)^P(\mathtt{Sid}_X) \ .$$

We shall check that all mappings hiding the bounded sets of bornology spaces are also hiding between siding spaces.

Let we have the point $y \in Y$ and set $A' \subset Y$ siding in the bornology space Y, i. e. we can find a bigger bounded set $A' \subset B'$ with taken point $y \in B'$. For hiding mapping we can find a bounded set $B \subset X$ with direct image $f^P(B) = B'$. Therefore we also can find a covering point $x \in X$ with $f(x) = y$. Taking the set $A := f^{-1}(A') \cap B$ we get needed siding set $f^P(A) = A'$.

In general we can't check that hiding mapping between siding spaces is hiding the bounded sets.

At first we notice that every bounded point $y \in Y$ must be covered. Such point has proximal set $A' \subset Y$, therefore for hiding mapping will be the point $x \in X$ with $f(x) = y$ and the set $A \subset X$ with $f^P(A) = B$.

Let the set $B' \subset Y$ is bounded for the siding space, i. e. it has a bigger siding set $B' \subset \langle B' \rangle$. Every point $y \in B'$ will have $x \in X$ with $f(x) = y$ and $B_x \subset X$ with $f^P(B_x) = B'$, therefore these points compounds the set $A \subset X$ with direct image $f^P(A) = B'$. We want to check that the new set has bigger siding set

$$A \subset \langle A \rangle$$

But it can't be calculated without some asumption of compactness for the union of proximal sets

$$\bigcup \{B_x : x \in X\} \; .$$

The reversely wasting (+13) mapping between siding spaces demands small inverse image of siding filter

$$(f^P)^{-1}(\mathcal{F}_y) \leq \mathcal{F}_{f^{-1}(y)} \; .$$

I don't think that these property may have global counterpart for bornology spaces.

At this moment I don't see any application of such continuity properties for bornological spaces. May be they become interesting for various final or initial constructions of siding spaces.

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